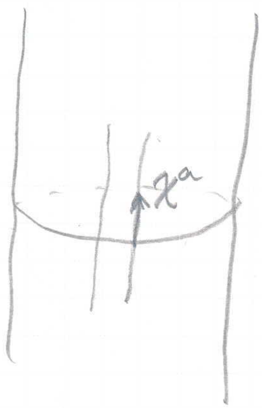
III. Black hole mechanics

1. Killing horizon, Surface gravity and area of the event horizon

* Killing horizon

An event horizon is called a Killing horizon when it possesses a Killing generator,

i.e., a null surface to which a Killing field χ^a is normal.



$$\Downarrow$$

$$\mathcal{L}_\chi g_{ab} = \nabla_a \chi_b + \nabla_b \chi_a = 0 .$$

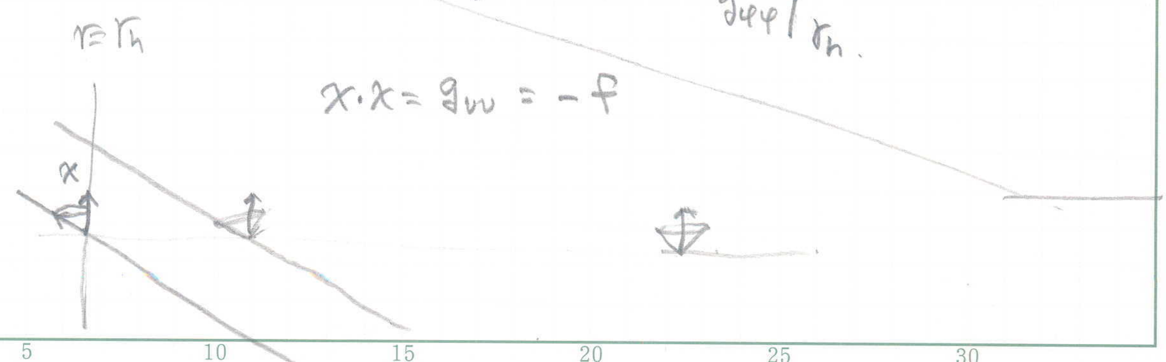
Ex) Schwarzschild b.h & R-N : $\chi^a = (\partial_t)^a$ in (t,r)

$= (\partial_v)^a$ in (v,r)

Kerr bh. $\chi^a = (\partial_t)^a + \Omega_H (\partial_\varphi)^a$

w/ $\Omega_H = - \left. \dfrac{g_{t\varphi}}{g_{\varphi\varphi}} \right|_{r_h}$.

$\chi \cdot \chi = g_{vv} = -f$



* Surface gravity

Since $\chi\cdot\chi = 0 = \text{const.}$ on the horizon and χ^a is normal to it, $\nabla^a(\chi\cdot\chi) \propto \chi^a$.

$$\Rightarrow \quad \nabla^a(\chi\cdot\chi) \equiv -2\kappa\,\chi^a \text{ at the horizon.}$$

$$\nabla_a(\chi^b\chi_b) = 2\chi^b\nabla_a\chi_b = -2\chi^b\nabla_b\chi_a$$

$$\boxed{\therefore \quad \chi\cdot\nabla\chi^a = \kappa\,\chi^a}$$

Geodesic equation in a non-affine parametrization.

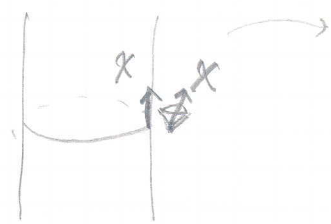
How to evaluate κ for a given horizon?

i) Use the formula above in the vicinity of the horizon by using a regular coordinate system and take the limit of $r \to r_h$.

Prob. 3-1 Do it for the R-N b.h. in the Edington-Finkelstein coordinate system (v, r, θ, φ).

Answer: $\kappa = \frac{1}{2}\left(\frac{df}{dr}\right)_{r_h}$.

Sheet ______ Date ______

ii) 

$$\chi \cdot \chi < 0$$

$$u^a \equiv \frac{\chi^a}{\sqrt{-\chi \cdot \chi}} \quad : \text{4-vector of a stationary observer.}$$

Acceleration : $a^b = u \cdot \nabla u^b = \dfrac{\chi \cdot \nabla \chi^b}{-\chi \cdot \chi}$.

One can show that

$$\kappa^2 = -\frac{1}{2} (\nabla^a \chi^b)(\nabla_a \chi_b) \Big|_{r \to r_h}.$$

$$= \lim_{r \to r_h} \frac{\chi \cdot \nabla \chi^c \, \chi \cdot \nabla \chi_c}{-\chi \cdot \chi}$$

$$\therefore \boxed{\kappa = \lim_{r \to r_h} (V a)}$$

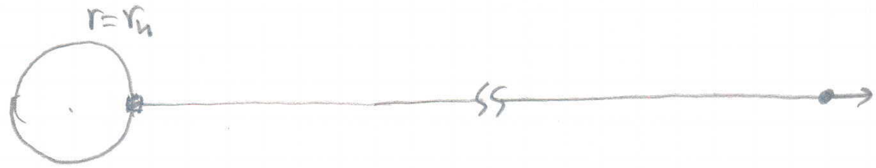
Here $a \equiv \sqrt{a^c a_c}$ & $V = \sqrt{-\chi \cdot \chi}$.

$\downarrow$ for a static b.h. in which $\chi \cdot \chi \to -1$ at $r = \infty$.

redshift factor

$\Rightarrow \kappa =$ "redshifted proper acceleration" of the orbits of χ^a near the horizon.

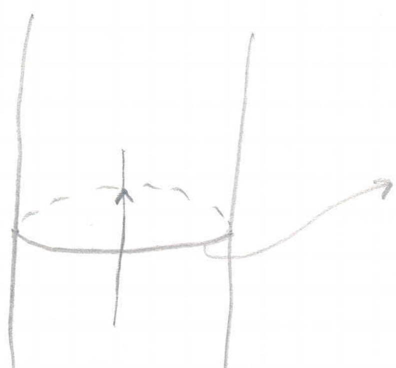
Or, force to be exerted at infinity in order to hold a unit test mass at the horizon.



$\frac{F_\infty}{m} = \kappa$.

<u>Prob. 3-2</u>. Solve the problem 4 in ch. 6 in Wald's.

* <u>Area of the horizon</u>.



2D spacelike hypersurface.

h_{ab} : induced metric

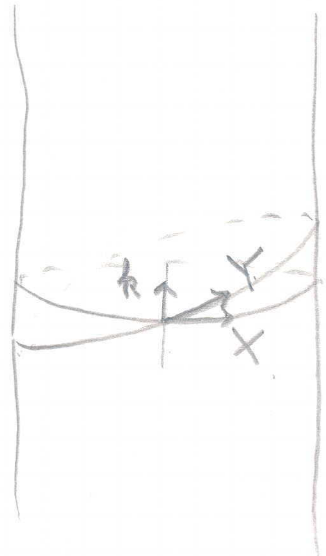
$$A \equiv \oint d^2x \sqrt{h}$$

Ex) $ds^2 = -f\,dv^2 + 2\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$\overset{r=r_h}{\Longrightarrow} r_h^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$$\therefore A = \int d\theta\, d\varphi \sqrt{g_{\theta\theta}\, g_{\varphi\varphi}} = \begin{cases} 4\pi r_h^2 & \text{for R-N bh.} \\ 4\pi(r_h^2 + a^2) & \text{for Kerr bh.} \end{cases}$$

Invariance of the value of the horizon area under Lorentz transformations :



k : null vector

$$Y = X + \alpha k .$$

$$Y \cdot Y = X \cdot X + 2\alpha \underbrace{X \cdot k}_{=0} + \alpha^2 \underbrace{k \cdot k}_{=0} .$$

$$\therefore Y \cdot Y = X \cdot X .$$

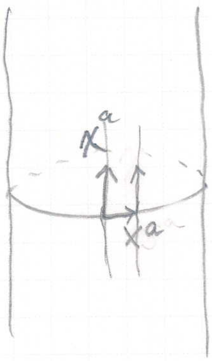
2. Four laws of black hole mechanics

	Black hole mechanics [redacted]	Ordinary θ-dynamics
-1^{st}	The event horizon of a stationary black hole must be a Killing horizon.	A thermal state is an equilibrium state.
0^{th}	$\kappa = const.$ over the horizon	$T = const.$ throughout a thermal body
1^{st}	$dM = \frac{1}{8\pi}\kappa\, dA + \Omega_H dJ + \Phi_H dQ$	$dE = TdS + pdV + \cdots$
	CONSERVATION LAW	
2^{nd}	$\delta A \geq 0$ in any process	$\delta S \geq 0$ for an isolated thermal system
3^{rd}	Impossible to achieve $\kappa = 0$ in a finite number of physical processes	$T \not\to 0$ ($S \to 0$ as $T \to 0$)

0th law of black hole mechanic

* [redacted]

$$\chi\cdot\nabla\chi_a = -\kappa\chi_a$$



i) $\kappa = \text{const.}$ along each orbit χ^a.

pf) $\mathcal{L}_\chi(\chi\cdot\nabla\chi_a) = -\mathcal{L}_\chi(\kappa\chi_a)$

右 $= -(\mathcal{L}_\chi\kappa)\chi_a - \kappa\underbrace{\mathcal{L}_\chi\chi_a}_{=0}$. ∵ $\mathcal{L}_v w_a = [v,w]_a = v\cdot\nabla w_a - w\cdot\nabla v_a$.

左 $= \underline{\chi^b\nabla_b(\chi\cdot\nabla\chi_a)} - \cancel{(\chi\cdot\nabla\chi^b)\nabla_b\chi_a}$

$= \cancel{\chi^b\nabla_b\chi^c\nabla_c\chi_a} + \chi^c\chi^b\nabla_b\nabla_c\chi_a$

$= \chi^c\chi^b\underline{\nabla_b\nabla_c\chi_a} = R_{acb}{}^{d}\chi_d$ (∵ χ^a is a Killing vector field $\nabla_a\chi_b + \nabla_b\chi_a = 0$)

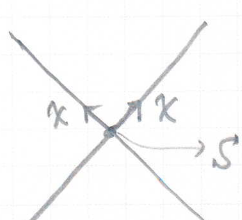
See p.442 in Wald's

$= \chi^c\underbrace{\chi^b\chi^d}_{\text{symm. in (bd)}}R_{ac\underbrace{bd}_{\text{anti-symm. in (bd)}}}$.

$= 0$.

$$\boxed{\therefore\ \mathcal{L}_\chi\kappa = \chi\cdot\nabla\kappa = 0}$$

ii) $\kappa = \text{const.}$ on a bifurcate Killing horizon.



A pair of null surfaces which are generated by a null Killing vector field χ orthogonal to a (D-2) spacelike pf) $\kappa^2 = -\frac{1}{2} \nabla_a \chi_b \nabla^a \chi^b$

$$\chi^c \nabla_c \kappa^2 = 2\kappa \chi^c \nabla_c \kappa = -\frac{1}{2}\left(\chi^c \underline{\nabla_c \nabla_a \chi_b}\right) \nabla^a \chi^b \times 2$$

$$= R_{bac}{}^{d} \chi_d$$

$$\Rightarrow \quad \kappa \chi \cdot \nabla \kappa = -\frac{1}{2} \underline{\chi_d}\, \chi^c R_{bac}{}^{d} \nabla^a \chi^b \longrightarrow 0 \quad \text{on } S.$$

Thus, κ is constant on the bifurcate surface S.

Since $\kappa = $ const. along each generator χ, it means that $\kappa = $ const. on the whole horizon.

iii) In GR, $\kappa = $ const. on any (not necessarily bifurcate) Killing horizon provided that the <u>dominant energy condition</u> is satisfied.

pf) See p. 332 in Wald's.

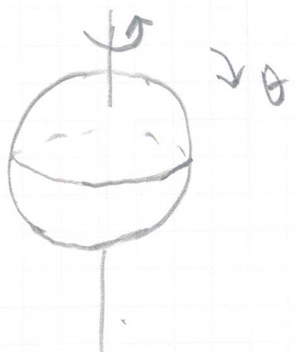
T_{ab}  k^a $k \cdot k < 0$

$\underbrace{-T_{ab} k^b}$: Energy-momentum 4-current density.

→ Future directed time-like or null vector for any future directed, timelike k^a.

Equivalently, $\boxed{T_{ab} v^a w^b \geq 0 \quad \forall \text{ future directed timelike}}$

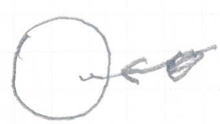
For the Schwarzschild bh, $\kappa = $ const. is trivial due to the spherical symmetry. In the case of the Kerr bh, however, this property is rather surprising since the curvature actually varies along the polar direction.



1st law of black hole mechanic

* [redacted]

This law states a sort of energy conservation. When a stationary black hole is perturbed by some matter, the changes of parameters characterizing such black hole hold some relationship so that the energy is conserved.



Ex) For Schwarzschild bh,

$$A = 4\pi r_h^2 = 4\pi (2M)^2 = 16\pi M^2.$$

$$\delta A = 32\pi M \delta M \Rightarrow \delta M = \frac{1}{8\pi} \frac{1}{4M} \delta A.$$

$$= \kappa \left(= \frac{M}{r_h^2} \right)$$

Sheet ______ Date ______

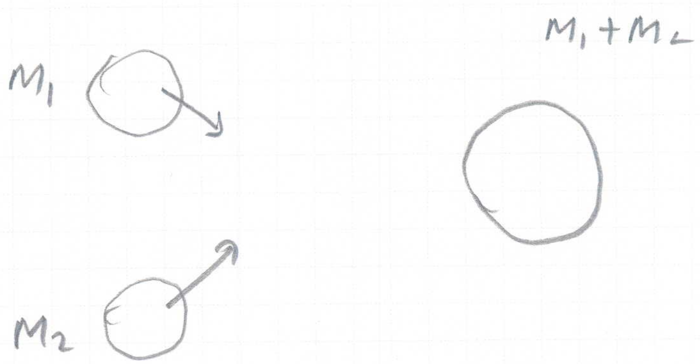
Note that $\delta M = \frac{1}{8\pi} \kappa \, \delta A + \Omega_H \, \delta J + \cdots$

ADM energy defined at infinity

Defined at the horizon.

2nd law of black hole mechani

* [redacted]



$A \sim M_1^2 + M_2^2$ $\qquad$ $A' \sim (M_1 + M_2)^2 = A + \underline{2M_1 M_2}$

$> A_1, A_2, A.$

<u>Black hole area theorem</u> : Hawking 1971

In GR, the area of the black hole event horizon never decrease provided that the <u>cosmic censorship conjecture</u> and the <u>null energy condition</u> hold.

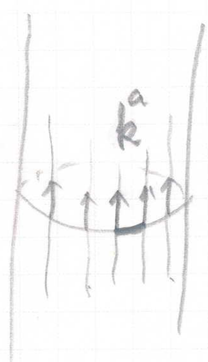
$T_{ab} k^a k^b \geq 0 \quad \forall$ null vectors k^a

→ No naked singularity

pf) $A = \oint d^2x \sqrt{h}$

$$\frac{dA}{d\lambda} = \oint d^2x \frac{d\sqrt{h}}{d\lambda} = \oint d^2x \sqrt{h}\left(\frac{d\sqrt{h}/d\lambda}{\sqrt{h}}\right) = \oint d^2x \sqrt{h}\,\underbrace{\theta}_{\text{Expansion}}.$$

It can be shown that $\theta \geq 0$ under the conditions described above.



For a congruence of null geodesics on the horizon,

shear twist

$$k\cdot\nabla\theta = \frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \underbrace{\hat{\sigma}_{ab}\hat{\sigma}^{ab}}_{\geq 0} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{ab}k^a k^b.$$

can be set to be zero

Raychaudhuri's eq.

≥ 0 since $\hat{\sigma}_{ab}$ is a spacelike tensor

$$\left(R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G\, T_{ab}\right)k^a k^b$$

$$\Rightarrow R_{ab}k^a k^b - \frac{1}{2}R\, \underbrace{k\cdot k}_{\to 0} = 8\pi G\, T_{ab}k^a k^b$$

Thus, $R_{ab}k^a k^b \geq 0$ if $\underbrace{T_{ab}k^a k^b \geq 0}_{\text{null energy condition}}$

Then $\frac{d\theta}{d\lambda}$

Raychaudhuri's eq. :

$$k\cdot\nabla\theta \equiv \frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} + \hat{\omega}_{ab}\hat{\omega}^{ab} - R_{ab}k^a k^b .$$

$\hat{\sigma}_{ab}$: shear, spacelike tensor $\Rightarrow$ $\hat{\sigma}_{ab}\hat{\sigma}^{ab} \geq 0$.

ω_{ab} : twist. it can be set to be zero.

Einstein's eq.: $R_{ab} - \frac{1}{2} R g_{ab} = 8\pi G T_{ab}$

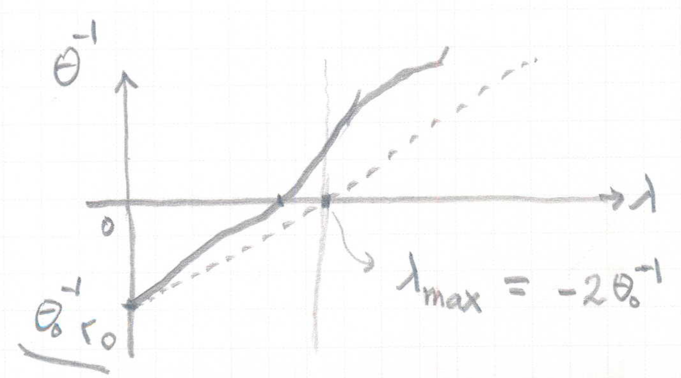
$$R_{ab}k^a k^b - \frac{1}{2} R \cancelto{0}{k\cdot k} = 8\pi G \underline{T_{ab}k^a k^b} \geq 0$$ if the null energy condition holds for matters.

$\Rightarrow$ $R_{ab}k^a k^b \geq 0$.

Then the RHS is negative-definite.

$\Rightarrow$ $\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2$ Or, $\frac{d\theta^{-1}}{d\lambda} \geq \frac{1}{2}$

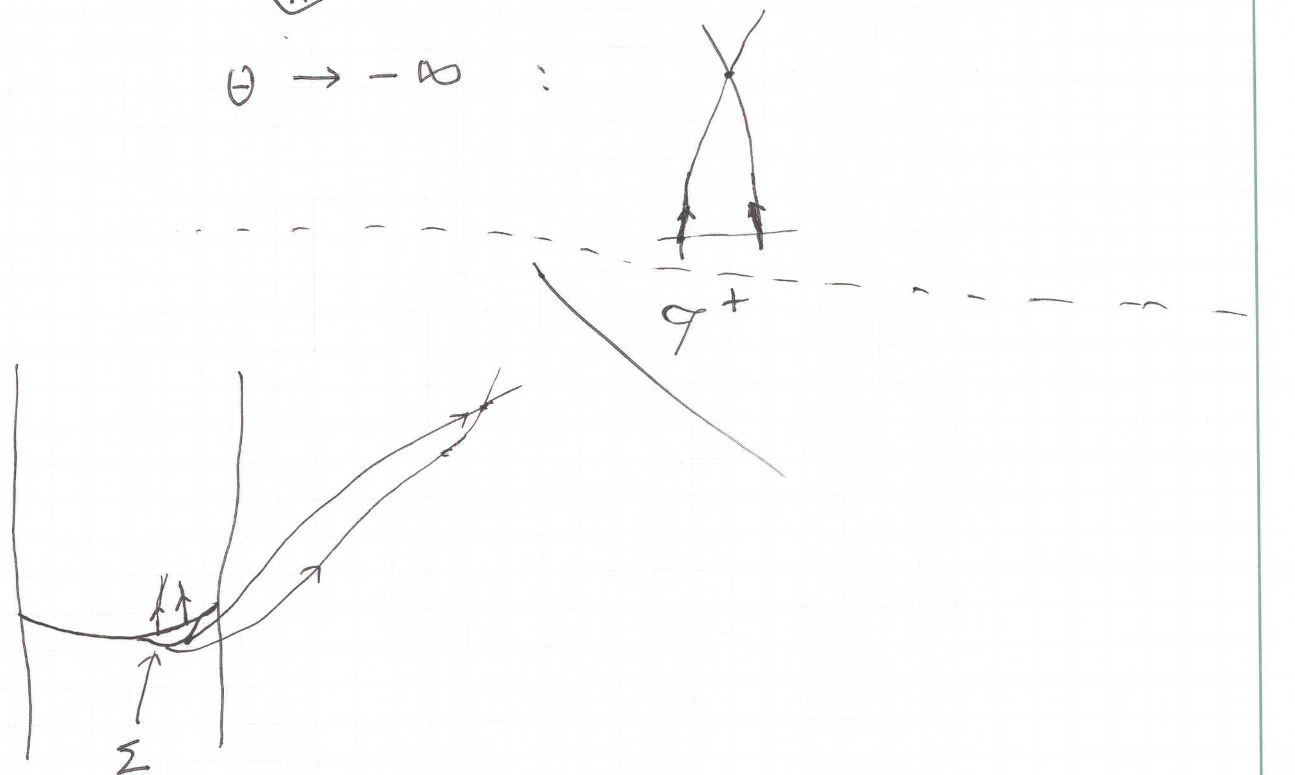
$\Rightarrow$ $\theta^{-1}(\lambda) \geq \frac{1}{2}\lambda + \theta_0^{-1}$.



Therefore, $\theta^{-1} \to 0$ in a finite affine time.

$\Updownarrow$

$\theta \to -\infty$:



Now suppose $\theta < 0$ at some point on the horizon. Then one can deform a spacelike slice of the horizon slightly outward to obtain a compact spacelike surface Σ so that $\theta < 0$ still everywhere on Σ. If cosmic censorship is assumed, there should be some null geodesic orthogonal to Σ that remains on the boundary of the future of Σ all the way out to $\mathscr{I}^+$. However, we have a contradiction because, as explained above, there should exist a conjugate point on its path, which implies it cannot stay on the boundary of $J^{+}(\Sigma)$.

therefore,

$$\theta \geq 0 \quad \text{at any point on the horizon.}$$

Namely,

$$\boxed{\frac{dA}{d\lambda} \geq 0 .}$$

To summarize, these properties of a stationary black hole are very much analogous to those of an ordinary thermodynamic system.

Is this simply an analogy, not more than that, or it indicates some deep physical meaning to be understood?

$\kappa \leftrightarrow T$: Black hole temperature?

$A \leftrightarrow S$: Black hole entropy?

WHAT DOES IT MEAN ???!!!

아! 저 나무 그늘
내 영혼 쉬어가기 좋겠네.