I. General properties of black holes in GR

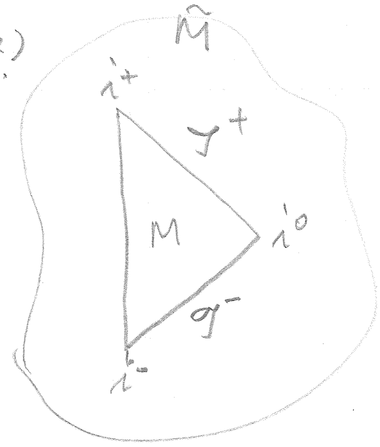
1. Definition of a black hole

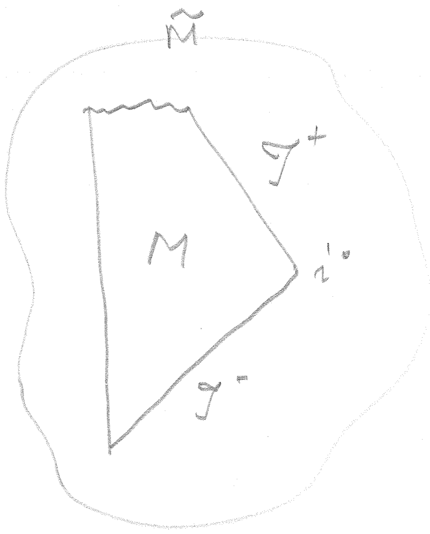
- "A region of spacetime from which nothing, not even light, can escape to infinity."
- It is usually formed through collapse of a star.

* <u>Asymptotically flat spacetime (M, g_{ab})</u>:

A spacetime which has the structure of the Minkowski flat ST at infinity. That is, $\exists\ (\tilde{M}, \tilde{g}_{ab})$ having a conformal isometry $\psi : M \to \psi[M] \subset \tilde{M}$ s.t. the boundary of M consists of i^0, $\mathscr{I}^+ = \dot{J}^+(i^0) - i^0$ and $\mathscr{I}^- = \dot{J}^-(i^0) - i^0$, and other suitable conditions are satisfied; see details in Wald (p. 276).

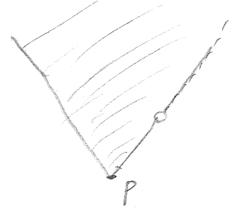
Ex)





i^0: spacelike infinity
$\mathscr{I}^+$: future null infinity
$\mathscr{I}^-$: past null infinity

- Causal future of $p \in M$:



$$J^+(p) = \{ q \in M \mid \exists \text{ a future directed causal curve } \lambda(t) \text{ with } \lambda(0) = p \text{ and } \lambda(1) = q \}.$$

* <u>strongly asymptotically predictable ST</u> :

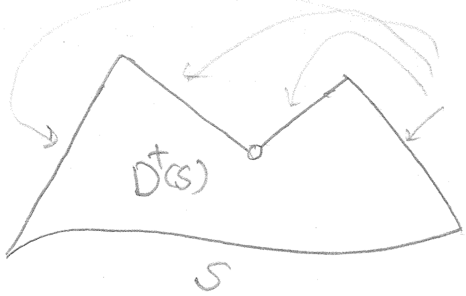
An asymptotically flat ST (M, g_{ab}) is said to be so if in the unphysical ST $(\tilde{M}, \tilde{g}_{ab})$ there is an open region $\tilde{V} \subset \tilde{M}$ with $\overline{J^-(\mathscr{I}^+) \cap M} \subset \tilde{V}$ s.t. $(\tilde{V}, \tilde{g}_{ab})$ is globally hyperbolic. Namely, $(\tilde{V}, \tilde{g}_{ab})$ possesses a Cauchy surface Σ,

↳ a spacelike surface

i.e., $D(\Sigma) = \tilde{V}$.

– Future domain of dependence of S :

$$D^+(S) = \{ p \in M \mid \text{Every past inextendible causal curve through } p \text{ intersects } S \}$$



$H^+(S)$: Cauchy horizon of S.

– $D(S) = D^+(S) \cup D^-(S)$.

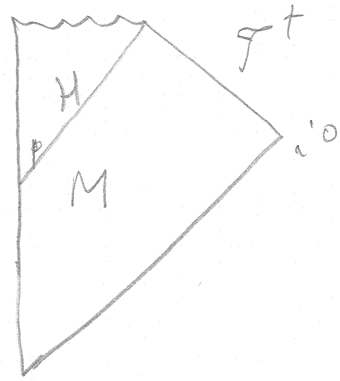
– A strongly asymptotically predictable ST M contains a "black hole" region if $M \not\subset J^-(\mathscr{I}^+)$.

* <u>Black hole region B</u> :

$$B = M - J^-(\mathscr{I}^+).$$

* Event horizon H :

$$H = \dot{J}(\mathscr{I}^{+}) \cap M .$$



Note : – The strong asymptotic predictability implies no "naked singularity" outside the black hole.

$\Rightarrow$ cosmic censorship conjecture.

– This definition of a black hole or an event horizon requires the full evolution of a ST.

$\Rightarrow$ "Global" in nature.

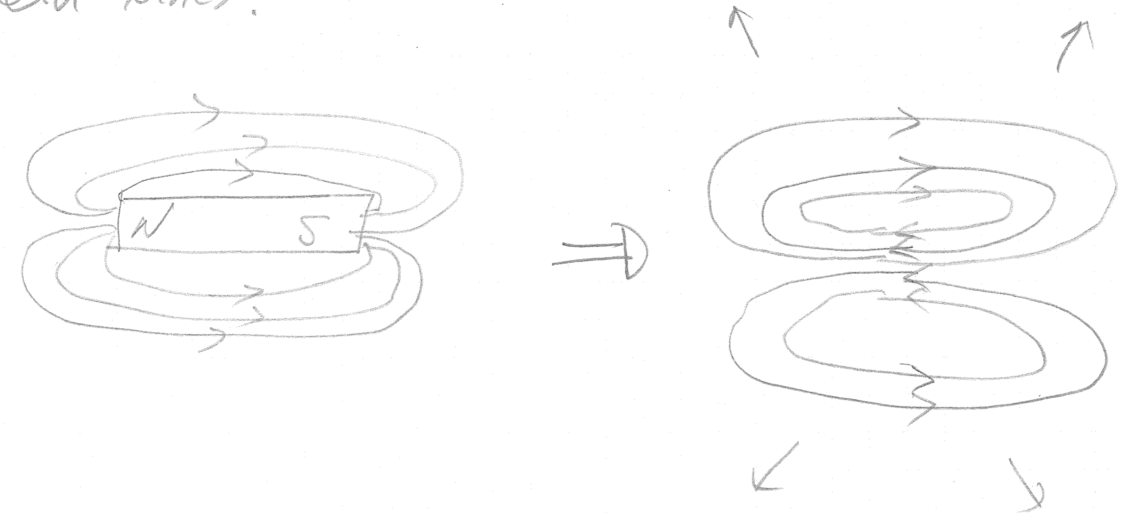
(See Ashtekar et al. for "local" definitions.)

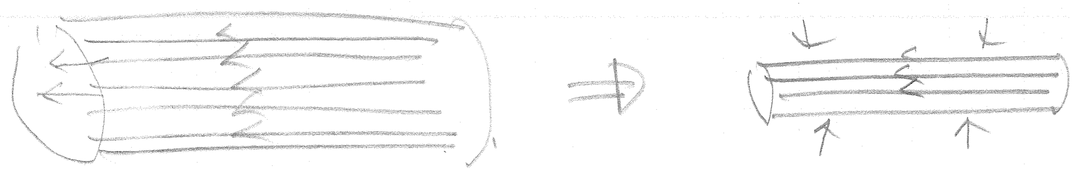
2. Hoop conjecture

– For a given matter distribution, when would a black hole be formed?

– How much compact should it be for the formation of a bh.?

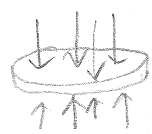
– In 1962 (?) Melvin found that an infinitely long cylindrical bundle of intense magnetic field lines could be confined since the field energy itself creates enough ST curvature (gravity) to hold the bundle together, despite the repulsion between field lines.





Implosion ?

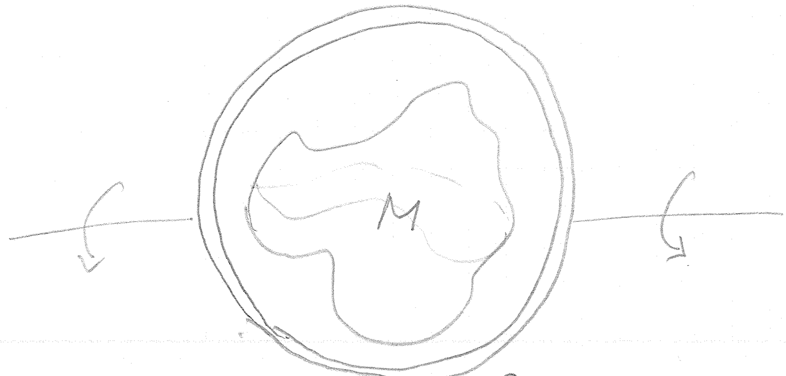
- Kip Thorne w/ Wheeler found No implosion.
- An object compressed in only two spatial directions
- If the object is compressed in only a single direction, into a very thin pancake, pressure will overwhelm gravity even more easily.



- Only if an object is compressed in all three spatial directions can gravity become so strong that it overwhelms all forms of internal pressure. ⇒ the case of sphere.

* Hoop conjecture : K. Thorne (72)

An object will form a black hole around itself when and only when a mass M gets compacted into a region whose circumference in EVERY direction in $C \leq 4\pi M$.



$C_{cr.} = 4\pi M$.

Note: - No precise mathematical formulation exists.

3. Singularity theorem
- In the spherical collapse of matter, a curvature singularity occurrs.
- Is this due to the spherical symmetry or generic?

Let (M, g_{ab}) be a connected globally hyperbolic ST with a noncompact Cauchy surface Σ. Suppose $R_{ab}k^a k^b \geqq 0$ for all null k^a, as will be the case if (M, g_{ab}) is a solution of Einstein's equation with matter satisfying the weak or strong energy condition. Suppose, further, that M contains a trapped surface T. Let $\theta_0 < 0$ denote the maximum value of θ for both sets of orthogonal geodesics on T. Then at least one inextendible future directed orthogonal null geodesic from T has affine length no greater than $\frac{2}{|\theta_0|}$.

* <u>Weak energy condition</u>:

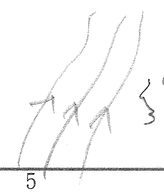
$$T_{ab}\xi^a\xi^b \geqq 0 \quad \text{for all timelike } \xi^a.$$

* <u>strong energy condition</u>:

$$T_{ab}\xi^a\xi^b + \frac{1}{2}T \geqq 0 \quad \text{for all unit timelike } \xi^a.$$

– Raychaudhuri eq. for a congruence of timelike geodesics

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - \underbrace{R_{ab}\xi^a\xi^b}$$



$$R_{ab} - \frac{1}{2}R g_{ab} = 8\pi T_{ab} \Rightarrow R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}T g_{ab}\right).$$

$$R - 2R = 8\pi T \qquad R = -8\pi T.$$

* <u>Dominant energy condition</u>:

For all future directed timelike ξ^a,

$-T^a{}_b \xi^b$: a future directed timelike or null vector.

↓

Energy-momentum 4-current density seen by ξ^a ⟹ The speed of energy flow of matter is always less than the speed of light.

Note: Dominant E.C. ⟹ Weak E.C.
Otherwise, they are independent.

* <u>Null energy condition</u>:

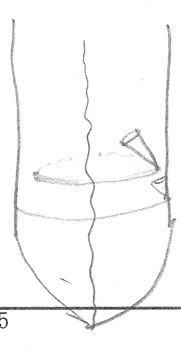
$T_{ab}\xi^a\xi^b \geq 0$ for all null ξ^a.

* <u>Trapped surface</u>:

A compact, two-dimensional, smooth spacelike submanifold T having the property that the expansion θ of future directed null geodesics orthogonal to T is everywhere <u>negative</u>.

↓

nonpositive.



* Marginally trapped surface

4. Cosmic censorship conjecture

* <u>Physical formulation</u>: All physically reasonable STs are globally hyperbolic, i.e., apart from a possible initial singularity (such as the "big bang" singularity) no singularity is never "visible" to any observer.

* <u>Precise formulation</u>: Let (M, g_{ab}) be an initial data set for Einsteins equation, with (Σ, h_{ab}) a complete Riemannian manifold and with the Einstein-matter equations with T_{ab} satisfying the dominant energy condition. Then, if the maximal Cauchy development of this initial data is extendible, for each $p \in H^+(\Sigma)$ in any extension, either strong causality is violated at p or $\overline{I(p) \cap \Sigma}$ is noncompact.

5. Topology theorem

The two-dimensional surface formed by the intersection of the horizon of a <u>stationary</u> bh with a Cauchy surface must have topology S^2.

<u>Note</u>. Black ring solution in 5D.

In 4D.

6. Uniqueness theorem

"The Kerr black holes are the only possible stationary vacuum black holes."

i) Hawking: A stationary vacuum bh must be static or axisymmetric.

ii) Israel ('67): The only black holes that are static, vacuum, topologically spherical are the Schwarzschild solutions.

iii) Carter ('71) & Robinson ('75) : All stationary axisymmetric black holes are uniquely characterized by two parameters which appear in the boundary conditions. Note that the Kerr solutions exhaust all possible values of these parameters.

Note that all these results have been generalized to the electrovac case easily.

⇒ The charged Kerr solutions are the only stationary, axisymmetric electrovac solutions.

* <u>Bekenstein ('72)</u> : Include other types of classical fields around a bh.

7. Apparent horizon

A marginally trapped surface having $\theta = 0$.

See the diagram in p. 321 by Hawking & Ellis.

8. No bifurcation theorem.

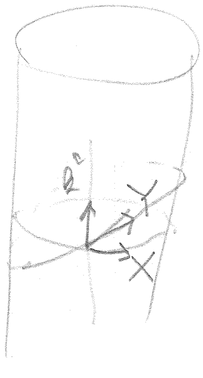
"A bh may never disappear nor may it "bifurcate", i.e., split into more than one bh at a later time."

Let (M, g_{ab}) be a strongly asymptotically predictable ST and let Σ_1 and Σ_2 be Cauchy surfaces for $\tilde{V}$ with $\Sigma_2 \subset I^+(\Sigma_1)$. Let $\mathcal{B}_1$ be a nonempty connected component of $B \cap \Sigma_1$, i.e., $\mathcal{B}_1$ is a black hole at time Σ_1. Then $J^+(\mathcal{B}_1) \cap \Sigma_2 \neq \phi$ and is contained within a single connected component of $B \cap \Sigma_2$.

(※ Go to the surface gravity)

9. Area theorem

* Lorentz invariance of the bh area



$$Y = X + \alpha k$$

$$Y \cdot Y = X \cdot X + 2\alpha \cancel{X \cdot k}^{\,0} + \alpha^2 \cancel{k \cdot k}^{\,0}$$

* 1st law $\Rightarrow$ 2nd law for quasistationary process.

(See the small note.)

* General proof : see the file (p.33 ~) .

10. surface gravity

* Definition : see p.24 ~ 27 in the file .