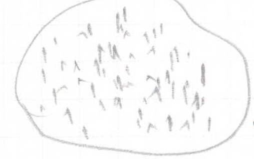
IV. Black hole entropy

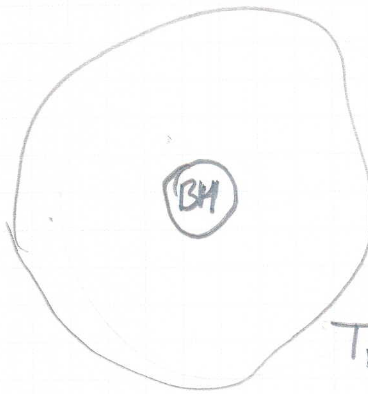
1. Black hole temperature

$$\delta E = T\delta S + P\delta V + \cdots$$



T

$$\delta M = \frac{1}{8\pi}\kappa\,\delta A + \Omega_H\,\delta J + \cdots$$



T_{BH}.

$E = M$: Indeed the total energy of the system.

$$\boxed{\begin{aligned} T_{BH} &= \#\,\kappa . \\ S_{BH} &= \#\,A . \end{aligned}}$$

Not known yet.

In Hawking radiation where a quantum matter is considered in a given black hole background, the particle emission rate is given by

$$\sim \frac{\Gamma(\omega)}{e^{2\pi\omega/\kappa}-1} . \quad \Leftrightarrow \quad \frac{1}{e^{E/kT}-1}$$

: Thermal black body radiation at temperature T.

$$\Rightarrow \boxed{kT_{BH} = \frac{\hbar\kappa}{2\pi c}} \quad \left(T_{BH} = \frac{\kappa}{2\pi} \quad w/\ \hbar=c=k=1 \right).$$

Ex) Schwarzschild bh. $\kappa = \frac{1}{4M}$

$$T = \frac{1}{8\pi M} = \frac{M_\odot}{M}\times 6\times10^{-8}\,{}^\circ K . \quad \downarrow \text{ as } \Delta M > 0$$

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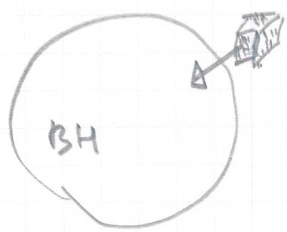
2. Bekenstein-Hawking entropy

* Bekenstein's conjecture



Without being assigned entropy to a black hole, one may easily construct a process in which

$$\Delta S_{total} < 0$$

$\Rightarrow$ Violation of the thermodynamic 2nd law ?

Bekenstein suggested that a black hole may possess entropy so that

$$\Delta S_{tot} = \Delta S_{SUROUND} + \Delta S_{BH} \geq 0 .$$

* Rough estimation of such BH entropy :

When a star collapses to form a black hole, all information of the star is hidden by the event horizon. Assuming one bit of information per subatomic particle,

$$S_{BH} \sim \text{Total information lost} \simeq \frac{M}{m_{subatom}}$$

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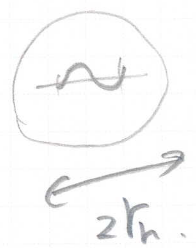
Classically, there would be no lower limit on the value of m. $S_{BH} \to \infty$ as $m \to 0$.

Being regarded as a quantum particle, however,



$$\lambda \sim \frac{\hbar}{mc^2} \lesssim 2r_h \sim M.$$

$$\therefore S_{BH} \sim \frac{M}{\hbar/M} \sim \frac{M^2}{\hbar} \sim \frac{A}{\hbar}.$$

$$dM = \underbrace{\frac{1}{8\pi}\kappa\,\delta A}_{} + \cdots$$

$$= \frac{\kappa}{2\pi}\cdot\delta\left(\frac{A}{4}\right).$$

$$\boxed{\therefore S_{BH} = \frac{A}{4G} = \frac{kc^3 A}{4G\hbar}}$$: Bekenstein-Hawking entropy.

$$S_{BH}^{\text{SOLAR}} \sim 10^{77} \gg S_{\text{SUN}} \sim 10^{58}$$

Enormously huge amount of entropy!

<u>Note</u>. In the 1st law of bh mechanics,

$$\delta M = \frac{\kappa}{2\pi} \cdot \delta\left(\frac{A}{4G}\right) + \cdots$$

NO $\hbar$ APPEARED! $\Rightarrow$ $(\hbar\kappa) \cdot \delta\left(\frac{A}{\hbar}\right)$.

$T_{BH} \sim \hbar\kappa \longrightarrow 0$ in the classical limit of $\hbar = 0$.

3. Euclidean path integral method

The partition function for a system in thermal equilibrium at temperature $T = \beta^{-1}$ is given by

$$Z = \mathrm{Tr}\, e^{-\beta H} = \sum_n e^{-\beta E_n} |\langle\phi_n|\phi\rangle|^2$$

$$= \sum_n \langle\phi|\phi_n\rangle \langle\phi_n| e^{-\beta H} |\phi_n\rangle\langle\phi_n|\phi\rangle$$

$$= \sum_{n,m} \langle\phi|\phi_n\rangle\langle\phi_n| e^{-\beta H} |\phi_m\rangle\langle\phi_m|\phi\rangle$$

$$= \langle\phi| e^{-\beta H} |\phi\rangle \qquad \beta = i(t_2 - t_1)$$

$$= \langle\phi| e^{-it_2 H} \cdot e^{it_1 H} |\phi\rangle$$

$$= \langle\phi(t_2)|\phi(t_1)\rangle.$$

$$= \int \mathcal{D}[\phi]\, e^{i S[\phi]}$$

$$t = -i\tau$$

$$= \int \mathcal{D}[\phi]\, e^{-S_E[\phi]} \quad \rightarrow \int_{\tau_1}^{\tau_2 = \tau_1 + \beta} d\tau\, \mathcal{L}_E[\phi]$$

$$\therefore\ Z = \int \mathcal{D}[\phi]\, e^{-S_E[\phi]}$$

where the integral is over all fields periodic in τ with periodicity β.

Similarly, one may obtain the partition function of a thermal system containg a black hole as follows;

$$Z = \int \mathcal{D}[g]\, e^{-I_E[g]}$$

$$I[g_{BH} + \delta g] = I[g_{BH}] + \cancelto{0}{I_1[\delta g]} + I_2[\delta g] + \cdots$$

$$Z = e^{-I_E[g_{BH}]} \cdot \int \mathcal{D}[\delta g]\, e^{-I_{2E}[\delta g] - \cdots}$$

$$\ln Z = \underbrace{-I_E[g_{BH}]}_{\text{Tree-level}} + \underbrace{\ln \int \mathcal{D}[\delta g]\, e^{-I_{2E}[\delta g]}}_{\text{one-loop}} + \cdots$$

$$I[g] = -\frac{1}{16\pi}\int_M (R + 16\pi \mathcal{L}_m) - \frac{1}{8\pi}\oint_{\partial M} K .$$

$$= -\int dt \left\{ \int_{\Sigma_t} \left[\pi^{ab}\dot{h}_{ab} \underbrace{- (N\mathcal{H} + N^a \mathcal{H}_a)\right] + \frac{1}{8\pi}\int_{S_t^\infty} (NK - N^a \pi_{ab} r^b)}_{H} \right\}$$

$$= -\int dt \left\{ \int_{\Sigma_t} \pi^{ab}\dot{h}_{ab} - H \right\}$$

For a stationary black hole, $\dot{h}_{ab} = 0$.

$$I_E[g_{BH}] = -\int d\tau \{ -H \} \text{ " } = \text{ "} \beta H \text{"}.$$

More precisely,

$$ds^2_{BH} = -f(r)\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 .$$

$$f(r) = \overset{0}{\overset{\shortparallel}{f(r_h)}} + f'(r_h)\cdot(r-r_h) + \cdots$$

$$t = -i\tau .$$

$$ds_E^2 = f(r)\,d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2 .$$

$r = r_h + \varepsilon$

$$\simeq f'\varepsilon\, d\tau^2 + \frac{d\varepsilon^2}{f'\varepsilon} + r_h^2 d\Omega^2 .$$

Recalling $\kappa = \frac{1}{2} f' \big|_{r=r_h}$.

$$ds_E^2 \simeq 2\kappa\varepsilon\, d\tau^2 + \frac{d\varepsilon^2}{2\kappa\varepsilon} + r_h^2 d\Omega^2 .$$

$$= d\rho^2 \qquad d\rho = \frac{1}{\sqrt{2\kappa}} \frac{d\varepsilon}{\sqrt{\varepsilon}} \quad \left(\frac{d\varepsilon}{\sqrt{\varepsilon}} = 2\, d\sqrt{\varepsilon}\right)$$

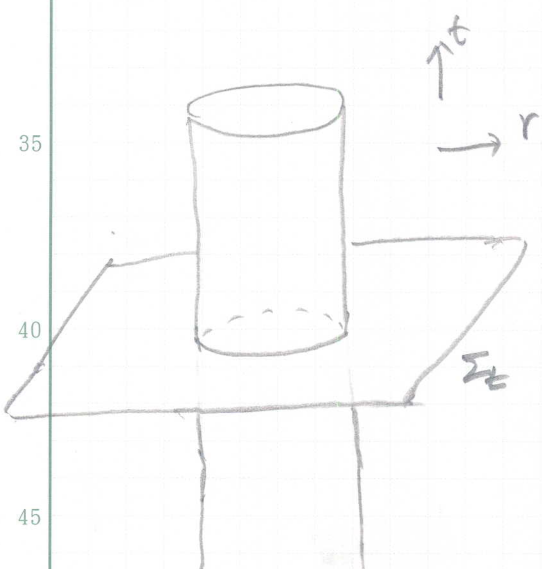
$$= d\sqrt{\frac{2\varepsilon}{\kappa}} .$$

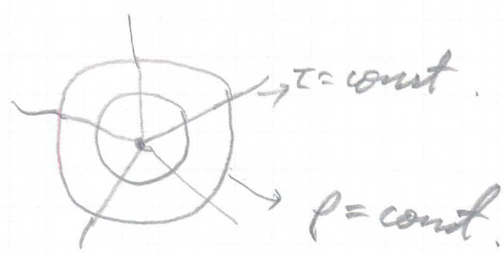
$$\therefore ds_E^2 \simeq \kappa^2 \rho^2 d\tau^2 + d\rho^2 + r_h^2 d\Omega^2 . \qquad \because \rho \equiv \left(\frac{2}{\kappa}\right)^{1/2} \sqrt{\varepsilon} .$$

$$= \underbrace{d\rho^2 + \rho^2 d(\kappa\tau)^2}_{} + \underbrace{r_h^2 d\Omega^2}_{} . \qquad \varepsilon = \frac{\kappa}{2}\rho^2 .$$

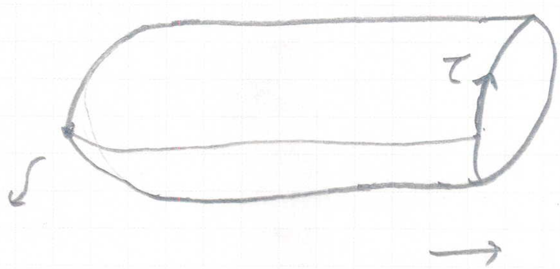
$$\mathbb{R}^2 \times S^2 . \Rightarrow \boxed{(\partial_\tau)^a : \text{Rotational Killing vector.}}$$

$\rho = 0 \iff$ Horizon.





$\Longleftrightarrow$



Lorentzian sector $\Leftarrow$ Euclidean sector

<u>Note</u>. i) No Euclidean region corresponding to the black hole region.

ii) In order for the Euclideanized BH to be regular, τ must be periodic with periodicity $2\pi/\kappa$.

Thus, it is natural to take $\beta = 2\pi/\kappa$,

i.e., $$T = \frac{\kappa}{2\pi}.$$

$T = \beta^{-1}$

Equilibrium when

$$T = T_{BH}.$$

$$\boxed{\therefore\ T_{BH} = \frac{\kappa}{2\pi} \quad \text{AGAIN!}}$$

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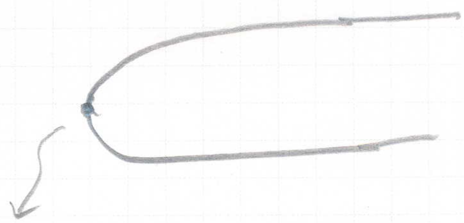
Actually, near the horizon <u>point</u>, the Euclidean geometry is

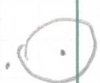


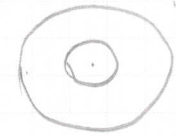
$\ast \otimes S^2$.

The contribution of this inner "boundary" to $I_E[g_{BH}]$ gives rise to

$$-\frac{1}{8\pi} \times 2\pi \chi \times A \qquad = -\frac{1}{4} A .$$

χ: Euler number $= \begin{cases} 1 & \text{for a disk} \\ 0 & \text{for an annulus} \end{cases}$





$$\therefore \ I_E = \beta H - \frac{1}{4} A , \qquad \ln Z = - I_E .$$

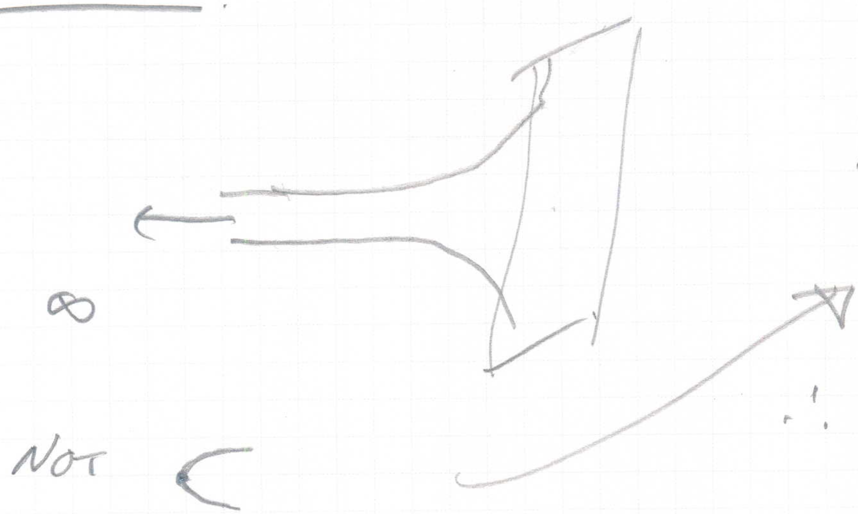
$$\langle E \rangle = - \frac{\partial}{\partial \beta} \ln Z = H = M_{ADM} .$$

$$S = -\left(\beta \frac{\partial}{\partial \beta} - 1\right) \ln Z = \frac{A}{4}$$

$$\boxed{\therefore \ S_{BH} = \frac{A}{4G} + \cdots\cdots}$$

Higher-order quantum corrections

REMARKS For extremal bhs,



No "$-\frac{1}{4}A$" term.

$\therefore\ S_{BH} = 0$.

even if $A \neq 0$.

Prob. 4-1 In R-N bh,

i) Show that the inner horizon coincides with the event horizon (i.e., $r_+ = r_-$) as $g \to M$.

ii) Such bh is called an extremal bh. Show that $\kappa \to 0$ for the extremal bh.

iii) Show that the proper distance between the extremal horizon and a certain point outside the event horizon is infinite.

iv) Check $S_{BH} = 0$ for extremal R-N bh by using the formulae explained above.

4. Noether charge method.

The well-known entropy-area relation $S_{BH} = A/4$ was obtained for a stationary black hole solution in Einstein gravity whose action is given by

$$I = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R + \mathcal{L}_{matter}(\phi, \nabla_a \phi, g_{ab}) \right].$$

Suppose the dynamics of gravitational fields is governed by more general action including higher-order derivative terms such as

$$I = \int d^Dx \sqrt{-g}\, \mathcal{L}(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \cdots, \nabla_{a_1} \cdots \nabla_{a_n} R_{bcde}, \phi, \nabla_a \phi, \cdots, \nabla_{a_1} \cdots \nabla_{a_m} \phi).$$

Does the entropy-area formula still hold for a stationary black hole solution in this theory?

This question has its own interest. In addition, note that the low energy effective action of some candidate theory such as string theory contains higher-order curvature interactions.

It is indicative to see that the area of the event horizon could decrease during the Oppenheimer-Snyder collapse in Brans-Dicke theories of gravity.

$$\mathcal{L}_{BD} = \frac{1}{16\pi G}\left(\phi R - \frac{\omega}{\phi}\nabla_a\phi\nabla^a\phi\right) + \mathcal{L}_{matter}.$$

(see Scheel, Shapiro & Teukolsky, PRD 51, 4236 (1995)

and G. Kang, PRD 54, 7483 (1996).)

since $$\nabla_a\nabla_b T^{cd\cdots} = \nabla_{(a}\nabla_{b)}T^{cd\cdots} + \underbrace{\nabla_{[a}\nabla_{b]}T^{cd\cdots}}_{= -R_{abe}{}^{c}T^{ed\cdots} - \cdots},$$

the most general covariant gravity action can be written as

$$I = \int d^Dx\sqrt{-g}\,\mathcal{L}(g_{ab}, R_{abcd}, \nabla_{a_1}R_{bcde}, \cdots, \nabla_{(a_1}\cdots\nabla_{a_n)}R_{bcde}, \phi, \nabla_a\phi, \cdots, \nabla_{(a_1}\cdots\nabla_{a_m)}\phi)$$

symmetrized covariant derivatives

Let $\mathbb{L} = \epsilon\mathcal{L}$: D-form

↳ D-dimensional volume element

$\sqrt{-g}\,dx^0\wedge dx^1\wedge\cdots\wedge dx^{D-1}$.

$$I = \int d^Dx\sqrt{-g}\,\mathcal{L} = \int \mathcal{L}\epsilon = \int \mathbb{L}$$

$$(g_{ab}, \phi) \to \psi$$

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$$\delta \mathbb{L} = \underbrace{\mathbb{E}\,\delta\psi} + d\,\Theta[\delta\psi] .$$

$$\left(= (\mathbb{E}_g)^{ab}\,\delta g_{ab} + \mathbb{E}_\phi\,\delta\phi \right)$$

$\Theta[\delta\psi]$: Symplectic potential $(D-1)$-form.
Dertermined only up to an additional closed $(D-1)$-form.

Let ξ^a be the infinitesimal generator of a diffeomorphism

For $\delta\psi = \pounds_\xi \psi$,

$$\delta_\xi \mathbb{L} = \pounds_\xi \mathbb{L} = \xi\cdot \underbrace{d\mathbb{L}}_{=0} + d(\xi\cdot\mathbb{L}) .$$

$$\therefore\quad d(\Theta[\delta\psi] - \xi\cdot\mathbb{L}) = -\mathbb{E}\,\pounds_\xi\psi \Rightarrow 0 \quad \text{on-shell.}$$

Thus, defining the Noether current $(D-1)$-form

$$\mathbf{j} \equiv \Theta[\pounds_\xi\psi] - \xi\cdot\mathbb{L} ,$$

we have $d\mathbf{j} = 0$ modulo the EOM, i.e., $\mathbf{j}$ is closed when the eqn. of motion is satisfied (i.e., $\mathbb{E}=0$). It implies

$\boxed{\mathbf{j} = d\mathbb{Q}}$ $\quad\mathbb{Q}$: Noether charge $(D-2)$-form associated with ξ^a

Consider an arbitrary variation $\delta\psi$ off of the solution ψ (i.e., $E(\psi)=0$).

$$\delta j = \delta\Theta[\psi, \pounds_\xi \psi] - \underbrace{\xi\cdot\delta L}$$

$$= \xi\cdot \underbrace{E}_{\substack{\| \\ 0}}\delta\psi + \underbrace{\xi\cdot d\Theta[\psi,\delta\psi]}$$

$$= \pounds_\xi \Theta[\psi,\delta\psi] - d(\xi\cdot\Theta[\psi,\delta\psi]).$$

$$\Rightarrow \delta j = \underline{\delta\Theta[\psi,\pounds_\xi\psi] - \pounds_\xi\Theta[\psi,\delta\psi]} + d(\xi\cdot\Theta[\psi,\delta\psi]).$$

Or, $\omega[\psi,\delta\psi,\pounds_\xi\psi] = \delta j - d(\xi\cdot\Theta[\psi,\delta\psi])$

where $\omega[\psi,\delta\psi,\pounds_\xi\psi] \equiv \delta\Theta[\psi,\pounds_\xi\psi] - \pounds_\xi\Theta[\psi,\delta\psi]$ is called the symplectic current $(D-1)$-form.

For any vector field ξ^a s.t. $\underline{\pounds_\xi\psi = 0}$ and the perturbation $\delta\psi$ satisfying the linearized eq., i.e.,

$$E[\psi+\delta\psi] = \underbrace{\overset{(0)}{E}(\psi)}_{\substack{\| \\ 0}} + \underbrace{\overset{(1)}{E}(\delta\psi)}_{\substack{\| \\ 0}} + \cdots,$$

one has $\omega[\psi,\delta\psi,\underline{\pounds_\xi\psi=0}] = 0$ and $\overset{\delta j}{\overset{\|}{\delta dQ}}[\psi,\pounds_\xi\psi] = d\,\delta Q$.

Then,

$$0 = d[\delta Q[\xi] - \xi \cdot \Theta[\psi, \delta\psi]]$$

Integrating it over a hypersurface C, we have

$$\oint_{\partial C} \delta Q[\xi] - \xi \cdot \Theta[\psi, \delta\psi] = 0 .$$

For a stationary black hole solution with bifurcate Killing horizon, let ξ^a be the Killing generator that vanishes on the bifurcation $(D-2)$-surface B.

$$\xi^a = (\partial_t)^a + \Omega_H^{(i)} (\partial_{\varphi_i})^a .$$

stationary Killing field

$\partial_t \cdot \partial_t \to -1$ at infinity

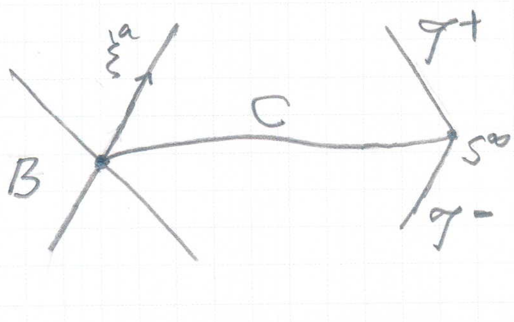
axial Killing fields

Absent for $D=2$

$i=1$ for $D=3,4$

$i=1,2,\dots$ for $D>5$.

Let C be an asymptotically flat spacelike hypersurface having B as its "interior boundary."



$$\int_B \left(\delta Q[\xi] - \xi \cdot \Theta[\psi, \delta\psi]\right)$$

$$= \int_{S^\infty} \left(\delta Q[\xi] - \xi \cdot \Theta[\psi, \delta\psi]\right)$$

Since $\xi^a \to 0$ on B, the left hand side becomes

$$\delta \int_B Q[\xi]$$

$\rightarrow$ Noether charge density associated with the symmetry generated by ξ^a.

Now suppose there exists a $(D-1)$-form $\mathbb{B}$ s.t.

$$\int_{S^\infty} \xi \cdot \Theta[\psi, \delta\psi] = \delta \int_{S^\infty} \xi \cdot \mathbb{B}[\psi]$$

Then, we have

$$\delta \int_B Q[\xi] = \delta \int_{S^\infty} \left(Q[\xi] - \xi \cdot \mathbb{B} \right)$$

$$= \delta \underbrace{\int_{S^\infty} \left(Q[\partial_t] - \partial_t \cdot \mathbb{B} \right)}_{= M} + \Omega_H^{(i)}\, \delta \underbrace{\int_{S^\infty} \left(Q[\partial_{\varphi_i}] - \partial_{\varphi_i} \cdot \mathbb{B} \right)}_{= -J_{(i)}}.$$

One can show that M: ADM mass (or total energy), $J_{(i)}$: angular momentum.

$$\boxed{\therefore\ \delta \int_B Q[\xi] = \delta M - \Omega_H^{(i)}\, \delta J_{(i)}. \qquad \text{w/ } \xi^a = (\partial_t)^a + \Omega_H^{(i)} \partial_{\varphi_i}}$$

One can show that

$$\delta \int_B Q[\xi] = \frac{\kappa}{2\pi} \delta \left(2\pi \int_B Q[\tilde{\xi}] \right).$$

pf) Since $\nabla_c \nabla_a \xi_b = -R_{abcd} \xi^d$ for a Killing vector ξ^a, one can always express $Q[\xi]$ as

$$Q[\xi] = \underbrace{\text{"}\xi^a\text{"}} + \text{"}\nabla_a \xi_b\text{"}.$$

No contribution on B since $\xi^a = 0$ there.

$$\delta \int_B Q[\xi] = \delta \int_B (X^{ab} \underbrace{\nabla_a \xi_b}_{= \kappa \epsilon_{ab}})$$

NEED MORE RIGOROUS PROOF.

$$= \kappa \, \delta \int_B X^{ab} \nabla_a \tilde{\xi}_b,$$

$\tilde{\xi}^a$: normalized to have unit surface gravity.

$$= \kappa \, \delta \int_B Q[\tilde{\xi}]$$

i.e., $\tilde{\xi}^b \nabla_b \tilde{\xi}_a = -1 \cdot \tilde{\xi}_a$.

Therefore, the black hole thermodynamic 1st law is established for a stationary bh solution in any arbitrarily diffeomorphism invariant gravity theory:

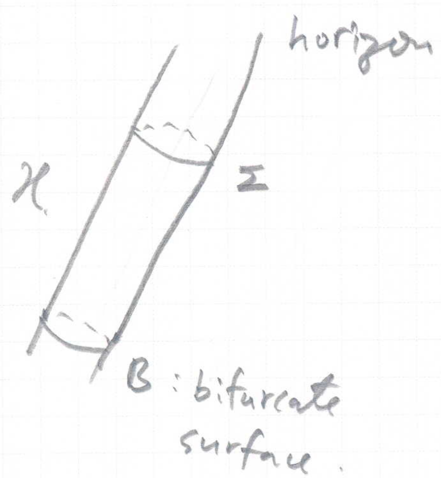
$$\boxed{\delta M = \kappa \, \delta \int_B Q[\tilde{\xi}] + \Omega_H^{(i)} \delta J_{(i)}.}$$

Since $T_{BH} = \kappa/2\pi$, it naturally gives the black hole entropy formular in general:

$$\boxed{S_{BH} = 2\pi \int_B Q}$$

Here, Q is the Noether charge density associated with the Killing field that is null on the horizon and normalized to have unit surface gravity.

For a stationary black hole horizon, it turns out that the Noether charge evaluated on the bifurcate surface is indeed same as that evaluated any other horizon cross section.



$$\int_{\mathcal{H}} dQ = \oint_B Q + \oint_\Sigma Q$$

$$\| $$

$$\int_{\mathcal{H}} j = \int_{\mathcal{H}} \Big(\cancel{\Theta[\psi, \underbrace{\mathcal{L}_\xi \psi}_{=0}]} - \xi \cdot L \Big) = 0$$

Since ξ^a is tangent on $\mathcal{H}$ and so the pull back of $\xi \cdot L$ vanishes on $\mathcal{H}$.

* Explicite formular for evaluating the Noether charge for a given arbitrary Lagrangian:

Ex) Einstein gravity: $\frac{1}{16\pi}\int d^4x \sqrt{g}\, R$.

$$\mathbb{L} = \frac{1}{16\pi}\epsilon R \qquad \left(\mathbb{L}_{abcd} = \frac{R}{16\pi}\epsilon_{abcd}\right)$$

$$\Theta_{abc} = \epsilon_{dabc}\frac{1}{16\pi} g^{de}g^{fh}(\nabla_f \delta g_{eh} - \nabla_e \delta g_{fh}).$$

$$\mathbb{J}_{abc} = \epsilon_{dabc}\frac{1}{8\pi}\nabla_e(\nabla^{[e}\xi^{d]})$$

$$\mathbb{Q}_{ab} = \epsilon_{abcd}\frac{-1}{16\pi}\underbrace{\nabla^c\xi^d}_{=\kappa\epsilon^{cd}} \qquad (\xi\cdot\nabla\xi^a = -\kappa\xi^a)$$

$$= \kappa\nabla^c\tilde{\xi}^d \quad \text{w/ } \tilde{\xi}^a = \kappa\xi^a$$

$$= \frac{1}{8\pi}\kappa\,\epsilon_{ab}.$$

$$\therefore S_{BH} = 2\pi\int \mathbb{Q}[\xi] = 2\pi\cdot\int\frac{1}{8\pi}\epsilon = \frac{1}{4}\int d^2x\sqrt{h} = \frac{A}{4}.$$

$$\frac{1}{16\pi}\int d^Dx\sqrt{g}\,\phi[R+\alpha R^2]$$

$$\Rightarrow S_{BH} = \frac{1}{4}\int d^{D-2}x\sqrt{h}\,\phi(1+2\alpha R)$$

In general,

$$\int d^D x \sqrt{-g}\left[R + F(\phi, R) + f(\phi) R_{\mu\nu}R^{\mu\nu} + \tilde{\mathcal{L}}(\phi, \nabla_a\phi, g_{ab}, R_{abcd}, \nabla_a R_{bcde}, \cdots) + \nabla^{2p}\phi\nabla^{2q}\phi\right]$$

$$S_{BH} = \frac{1}{4}\oint d^{D-2}x\sqrt{h}\left[1 + \frac{\partial F}{\partial R} + f(\phi)\, g_\perp^{\mu\nu} R_{\mu\nu} - \frac{1}{2}\left(Y^{abcd} - \nabla_e Z^{e:abcd}\right)\epsilon_{ab}\epsilon_{cd} + 0\right]$$

$$Y^{abcd} \equiv \frac{\partial \tilde{\mathcal{L}}}{\partial R_{abcd}}, \quad Z^{e:abcd} \equiv \frac{\partial \tilde{\mathcal{L}}}{\partial \nabla_e R_{abcd}},$$

$$g_\perp^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} = k^\mu l^\nu + k^\nu l^\mu$$

$l\cdot l = 0 = k\cdot k$

$k\cdot l = 1.$

Or,

$$\mathcal{L} = \mathcal{L}(g_{ab}, R_{abcd}, \nabla_{a_1}R_{bcde}, \cdots, \nabla_{(a_1}\cdots\nabla_{a_m)}R_{bcde}, \phi, \nabla_{a_1}\phi, \cdots, \nabla_{(a_1}\cdots\nabla_{a_n)}\phi)$$

$$E^{abcd} \equiv \frac{\partial\mathcal{L}}{\partial R_{abcd}} - \nabla_{a_1}\frac{\partial\mathcal{L}}{\partial(\nabla_{a_1}R_{abcd})} + \cdots + (-1)^m\nabla_{(a_1}\cdots\nabla_{a_m)}\frac{\partial\mathcal{L}}{\partial(\nabla_{(a_1}\cdots\nabla_{a_m)}R_{abcd})}.$$

$$\Rightarrow S_{BH} = 2\pi\int_\Sigma \mathbf{X}^{cd}\epsilon_{cd}$$

$\epsilon_{cd} \longrightarrow$ binormal to Σ

$$(\mathbf{X}^{cd})_{e_3\cdots e_D} = -E^{abcd}\epsilon_{abe_3\cdots e_D}.$$

Refs. Iyer & Wald, gr-qc/9403028, Jacobson, Kang & Myers, PRD 49, 6587 (1994).

BLACK HOLES

MASS	RADIUS	TEMPERATURE	ENTROPY	LUMINOSITY	LIFETIME
M	$2M$ $2\left(\frac{M}{M_{pl}}\right)\frac{\hbar}{M_{pl}c}$	$\frac{1}{8\pi M}$ $\frac{1}{8\pi}\left(\frac{M_{pl}}{M}\right)\frac{M_{pl}c^2}{k}$	$4\pi M^2$ $4\pi\left(\frac{M}{M_{pl}}\right)^2$	$\sim 10^{-3}/M^2$ $10^{-3}\left(\frac{M_{pl}}{M}\right)^2\frac{M_{pl}c^2}{t_{pl}}$	$\sim 10^3 M^3$ $10^3\left(\frac{M}{M_{pl}}\right)^3 t_{pl}$
$M_\odot \approx 10^{57}\,\frac{GeV}{c^2}$ $\approx 2\times10^{33}$ g	10^6 cm	10^{-7} °K	10^{77}	10^{-20} erg s^{-1}	10^{74} s
$M_E \approx 3\times10^{51}\,\frac{GeV}{c^2}$ $\approx 6\times10^{27}$ g	3 cm	10^{-1} °K	10^{66}	10^{-9} erg s^{-1}	10^{57} s
10^{26} g $\approx 10^{-7} M_\odot$	10^{-1} cm	3° K	10^{63}	10^{-6} erg s^{-1}	10^{53} s
10^{14} g	10^{-14} cm	10^{12} °K	10^{39}	10^{18} erg s^{-1}	10^{17} s
10^{7} g $= 10^4$ kg ≈ 10 tons	10^{-21} cm	10^{19} °K	10^{25}	10^{33} erg s^{-1}	10^{-4} s
$M_p \approx 1$ GeV/c^2 $\approx 2\times10^{-24}$ g	10^{-51} cm	10^{51} °K	10^{-37}	10^{94} erg s^{-1}	10^{-97} s
		Background radiation 3° K		Luminosity of sun 10^{33} erg s^{-1}	Age of universe 10^{17} s

$$M_{pl} = \sqrt{\frac{\hbar c}{G}} \approx 10^{19}\,\frac{GeV}{c^2} \approx 10^{-5}\,g \;;\quad M_{pl}c^2 \approx 10^{16}\,erg \;;\quad \frac{M_{pl}c^2}{k} \approx 10^{32}\,°K$$

$$\frac{M_{pl}c^2}{t_{pl}} \approx 10^{59}\,erg\,s^{-1} \;;\quad t_{pl} = \frac{\hbar}{M_{pl}c^2} \approx 10^{-43}\,s \;;\quad \frac{\hbar}{M_{pl}c} \approx 10^{-33}\,cm$$