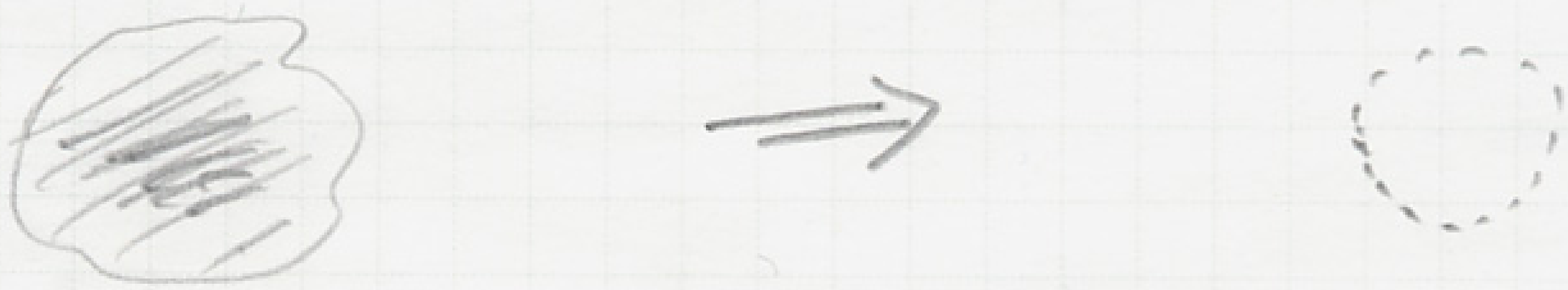
Introduction to the Hartle-Hawking state

— Lecture at Seogang U.

- QFT in a collapsing black hole
- QFT in an eternal bh ; $|0\rangle_K$ & $|0\rangle_S$.
- QFT in Rindler spacetime ; $|0\rangle_M$ & $|0\rangle_R$.
- Hartle-Hawking state

1. QFT in a collapsing black hole

ST with dense matter   stationary bh ST.



$|\psi\rangle_{t=-\infty}$ $|\psi\rangle_{t=\infty} = ?$

Especially, consider a state $|0\rangle$ which was "vacuum" at the past infinity, and see what happens to this state at the future infinity.

It turns out that this state is NOT 'vacuum' any more, but possesses 'particles' in it.

⇒ Hawking radiation! (1975).

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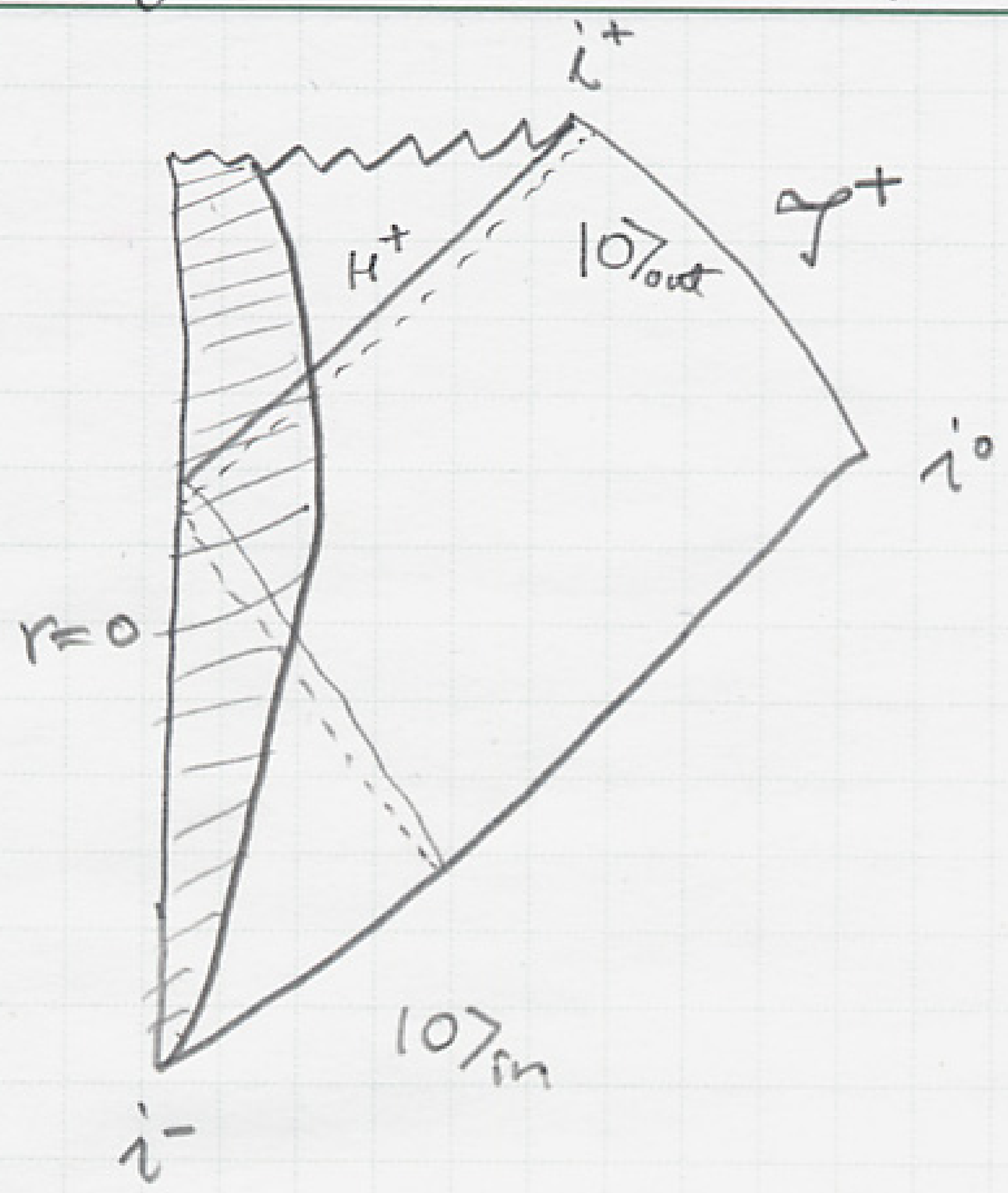
Penrose diagram for a collapsing neutral black hole



The state $|0\rangle_{in}$ possesses particles with respect to $|0\rangle_{out}$. Moreover, $|0\rangle_{in}$ behaves like a thermal state with temperature $T = \kappa/2\pi$ where κ is the surface gravity of the event horizon at the future infinity.

Note that the essential features of the Hawking radiation are independent of the details of collapse. This effect indeed turns out to be a consequence of the causal and topological structure of the black hole spacetime.

Therefore, we consider QFT in an eternal black hole spacetime which is an analytic extension of the late part of a collapsing black hole.

2. QFT in an eternal black hole background

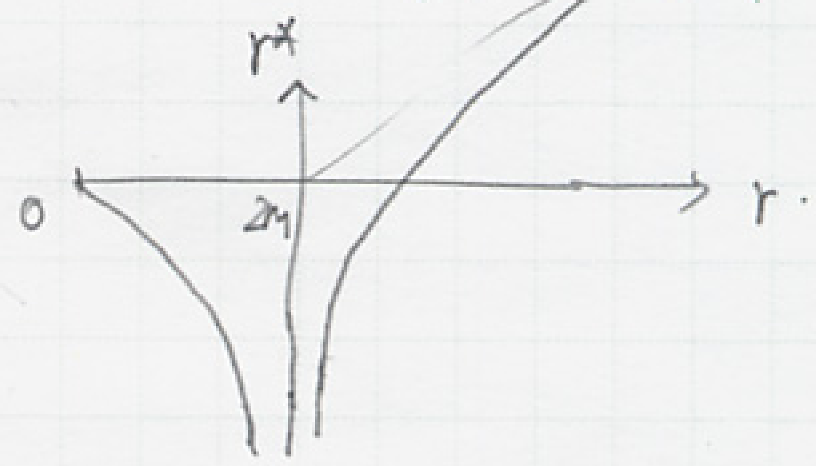
$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2 d\Omega^2 .$$

$$\begin{cases} t : -\infty \sim \infty \\ r : 0 \sim \infty \end{cases}$$

$$= -\left(1-\frac{2M}{r}\right)\left[dt^2 - \left(\frac{dr}{1-2M/r}\right)^2\right]$$

$$= -\left(1-\frac{2M}{r}\right)(dt - dr^*)(dt + dr^*) , \quad dr^* = \frac{dr}{1-2M/r}$$

$$= -\left(1-\frac{2M}{r}\right) du\, dv \qquad r^* = r + 2M \ln\left|\frac{r}{2M} - 1\right| .$$



$$\begin{cases} u = t - r^* & : \text{outgoing null coordinate} \\ v = t + r^* & : \text{ingoing null } \text{''} \end{cases}$$

Consider a massless scalar field

$$\Box\phi = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\, g^{\mu\nu}\partial_\nu\phi\right) = 0 .$$

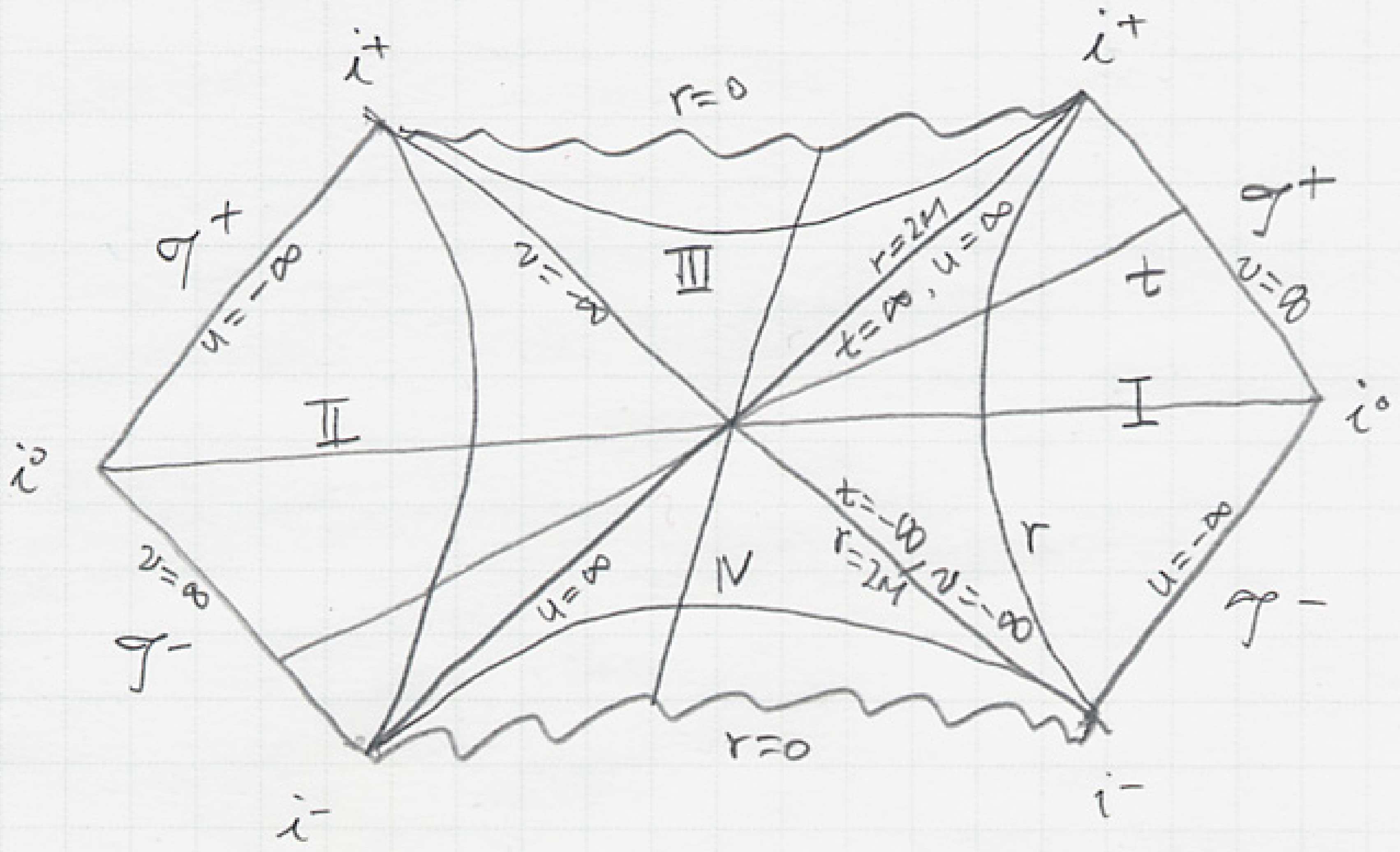
$$\Rightarrow \partial_u \partial_v \phi = 0 .$$

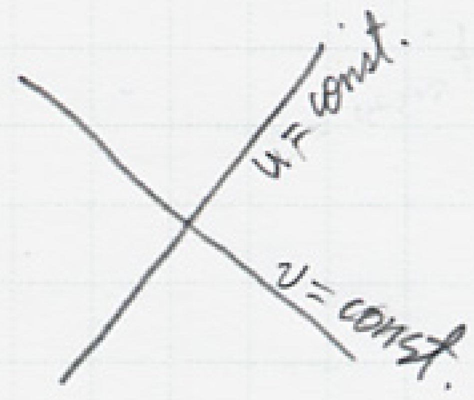
$$\phi \sim \begin{cases} e^{-i\omega u} = e^{-i\omega(t-r^*)} \overset{r\to\infty}{\sim} e^{-i\omega(t-r)} & : \text{Outgoing mode} \\ e^{-i\omega v} = e^{-i\omega(t+r^*)} \sim e^{-i\omega(t+r)} & : \text{ingoing mode} \end{cases}$$

These mode solutions are standard modes at $\mathcal{J}^+$, i.e., positive frequency modes with respect to the timelike Killing vector ∂_t

$$\mathcal{L}_{\partial_t}\phi = (\partial_t)^\mu \nabla_\mu \phi = \partial_t \phi = -i\omega\phi .$$

Penrose diagram for an eternal black hole





* Kruskal coordinates: (Kruskal 1960, Szekeres 1960);

$$ds^2 = -\left(1-\frac{2M}{r}\right) du\, dv$$

The metric is still singular in (u,v) coordinate system at $r(u,v)=0$ & $2M$.

Taking a coordinate transformation for the region I given by

$$\begin{cases} \bar{u} = -4M e^{-u/4M} & ; (-\infty, 0) \quad \text{for } u: (-\infty, \infty) \\ \bar{v} = 4M e^{v/4M} & ; (0, \infty) \quad \text{for } v: (-\infty, \infty) \end{cases}$$

$$\Rightarrow ds^2 = -\frac{2M}{r} e^{-r/2M} d\bar{u}\, d\bar{v}$$

Regular at $r(\bar{u},\bar{v}) = 2M$, but still singular at $r(\bar{u},\bar{v}) = 0$.

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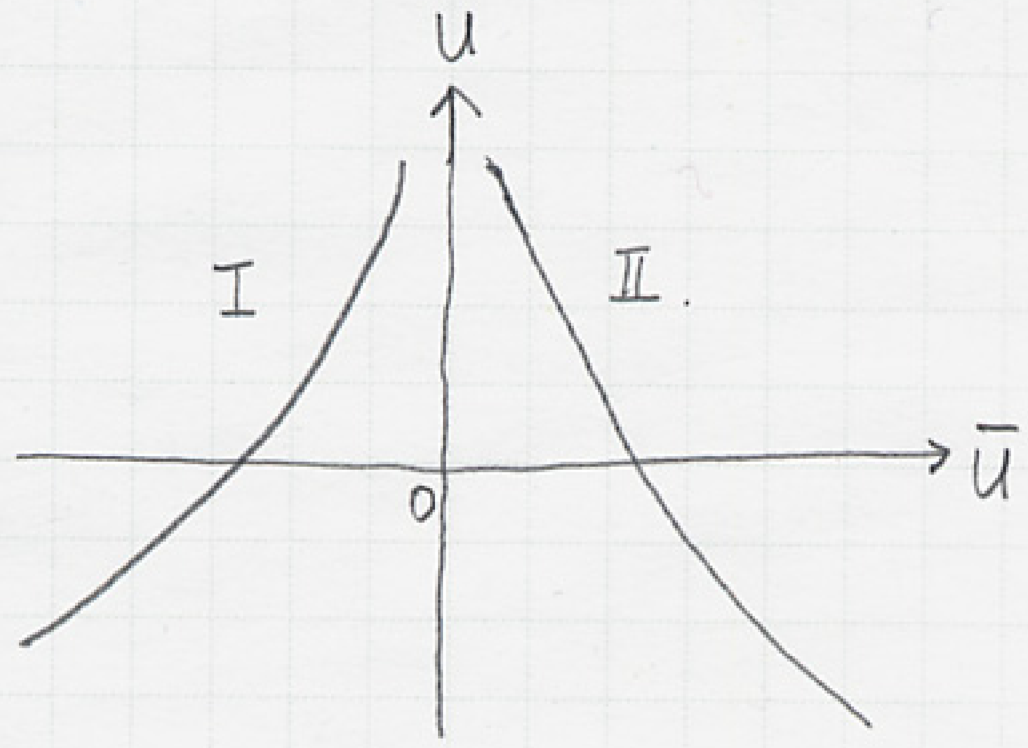
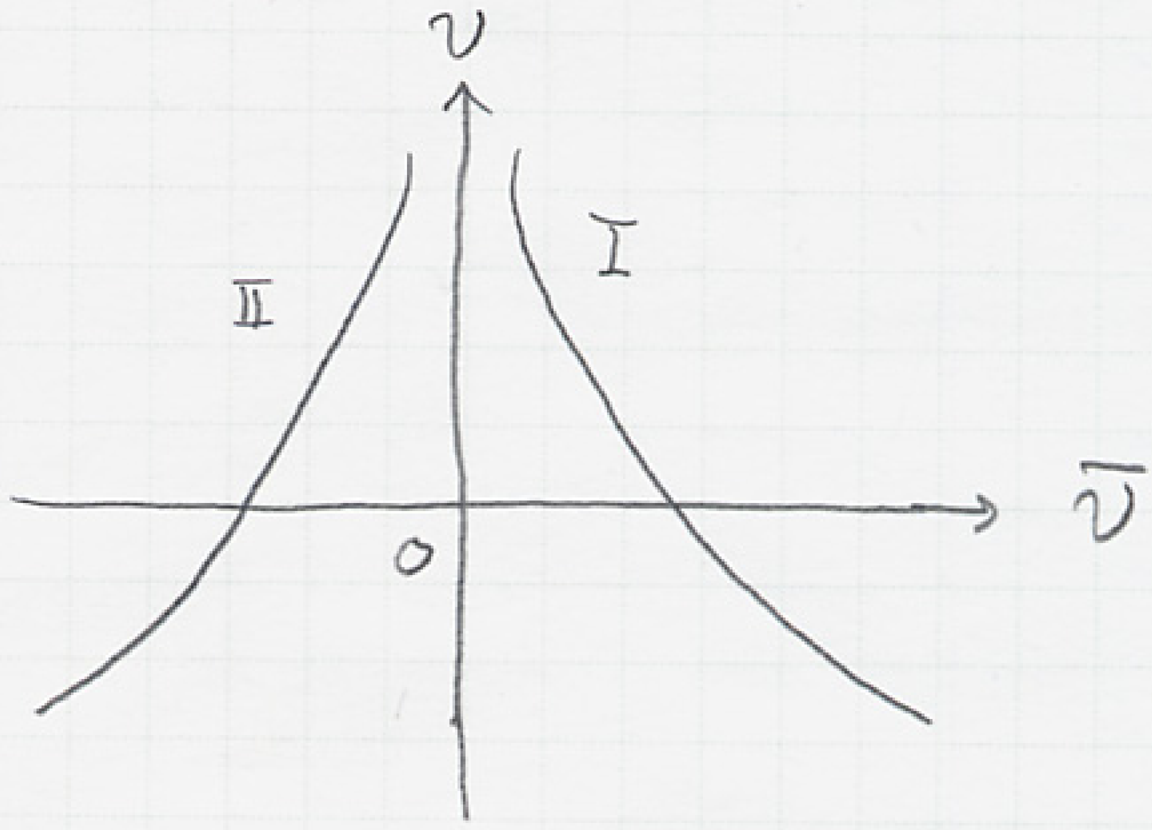
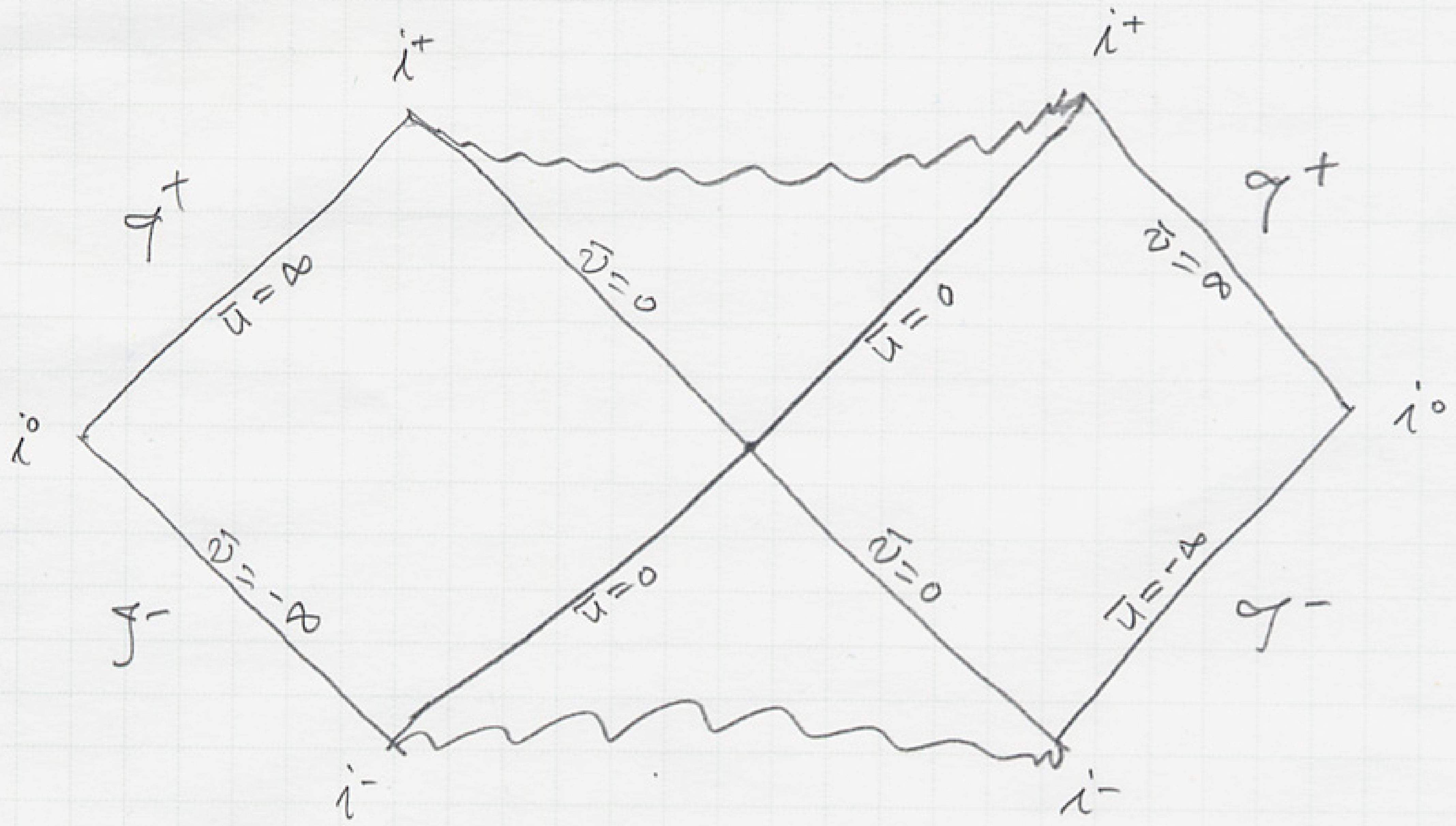
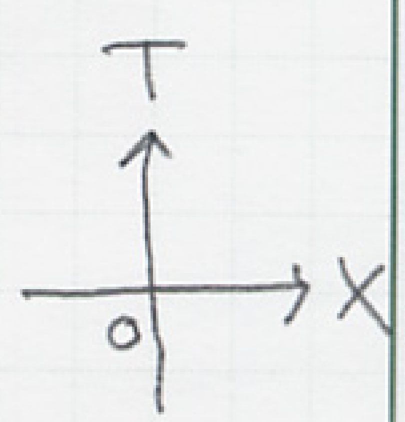
For the region II,

$$\begin{cases} \bar{u} = +4M e^{-u/4M} & ; \ (\infty, 0) \quad \text{for } u : (-\infty, \infty) \\ \bar{v} = -4M e^{v/4M} & ; \ (0, -\infty) \quad \text{for } v : (-\infty, \infty) \end{cases}$$









$$\begin{cases} T = \frac{1}{2}(\bar{u} + \bar{v}) & ; \ (-\infty, \infty) \\ X = \frac{1}{2}(\bar{v} - \bar{u}) & ; \ (-\infty, \infty) \end{cases} \quad \text{or} \quad \begin{cases} \bar{u} = T - X \\ \bar{v} = T + X \end{cases}$$

$$\Rightarrow ds^2 = \frac{2M}{r} e^{-r/2M} (-dT^2 + dX^2)$$

$(\bar{u}, \bar{v})$ or (T, X) : Kruskal coordinates.

$$(t,r) \to (u,v) \to (\bar{u},\bar{v}) \to (T,X)$$

$$\begin{cases} u = t - (r + 2M\ln|r/2M - 1|) \\ v = t + r + 2M\ln|r/2M - 1| \end{cases} \qquad \begin{cases} \bar{u} = \mp 4M e^{-u/4M} \\ \bar{v} = \pm e^{v/4M} \end{cases} \qquad \begin{cases} T = \frac{1}{2}(\bar{v} + \bar{u}) \\ X = \frac{1}{2}(\bar{v} - \bar{u}) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} t = \frac{1}{2}(v+u) = \frac{1}{2}\left[4M\ln(\pm\bar{v}) - 4M\ln(\mp\bar{u}/4M)\right] = 2M\ln\left(-4M\frac{\bar{v}}{\bar{u}}\right) = 2M\ln\left(\frac{X+T}{X-T}\right) + 2M\ln 4M \\ r + 2M\ln\left|\frac{r}{2M} - 1\right| = \frac{1}{2}(v-u) = 2M\ln\left(-\frac{\bar{v}\cdot\bar{u}}{4M}\right) = 2M\ln(X^2 - T^2) - 2M\ln 4M. \end{cases}$$

$$\frac{\bar{u}}{4M} \to \bar{u}$$

$$\Rightarrow \left(\frac{r}{2M} - 1\right)e^{r/2M} = X^2 - T^2$$

$$\frac{t}{2M} = \ln\frac{X+T}{X-T} = 2\tanh^{-1}\left(\frac{T}{X}\right).$$

$X = T$ at H.

Near the horizon, $r(T,X) = 2M + \mathcal{O}(X^2 - T^2)$

$$ds^2 \simeq e^{-1}(-dT^2 + dX^2) + (2M)^2 d\Omega^2.$$

$\therefore$ ∂_T is a Killing vector for $r \simeq 2M$.

Thus, $\phi \sim e^{-i\omega\bar{u}}$, $e^{-i\omega\bar{v}}$ are positive frequency modes with respect to ∂_T.

$$\mathcal{L}_{\partial_T}\phi = -i\omega\phi.$$

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Two natural basis modes ;

$\sim \{e^{-i\omega u}, e^{-i\omega v}\} \Rightarrow |0\rangle_S$: "Schwarzschild vacuum"

$\sim \{e^{-i\omega \bar{u}}, e^{-i\omega \bar{v}}\} \Rightarrow |0\rangle_K$: "Kruskal vacuum"

$\downarrow$

Hartle-Hawking state.

	H	$r=\infty$
t :	BAD	GOOD
T :	GOOD	

$\left\{ \begin{array}{l} u, v : \text{Schwarzschild null coordinates} \\ \bar{u}, \bar{v} : \text{Kruskal} \quad " \quad " \end{array} \right.$

$$\phi \sim e^{-i\omega u} = e^{-i\omega(t-r^*)}$$

$$\frac{\partial\phi}{\partial t} \sim \omega\phi .$$

↳ frequency.

$$= e^{-i\omega(-1)4M\ln(-\bar{u}/4M)}$$

$$\sim e^{+i\omega 4M \ln(T-X)}$$

$$\frac{\partial\phi}{\partial T} \sim \frac{\omega 4M}{T-X}\phi$$

$\swarrow$

∞ at the horizon.

$\therefore \{e^{-i\omega u}, e^{-i\omega v}\}$

oscillate infinitely rapidly on the horizon.

However, $\{e^{-i\omega\bar{u}}, e^{-i\omega\bar{v}}\}$ are regular there. $\frac{\partial\phi}{\partial T} \sim \omega\phi$.

We want to quantize ϕ in the extended Schwarzschild spacetime, i.e., Kruskal extension.

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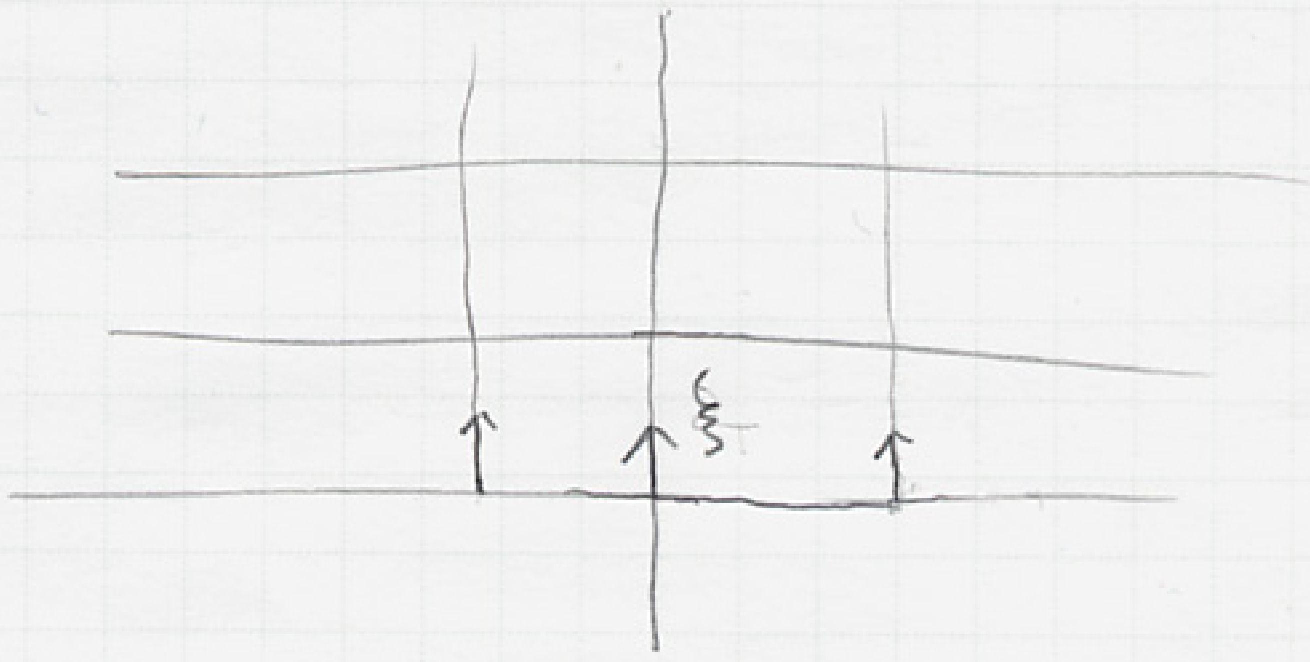
3. QFT in Rindler spacetime

The Kruskal extension turns out to be remarkably similar to the extension of the Rindler spacetime in its causal and topological structure.

$$ds^2 = -dT^2 + dX^2 \quad , \quad T : (-\infty, \infty) \quad X : (-\infty, \infty) .$$

In this two-dimensional flat spacetime, we have a time translation Killing vector field $\xi^a = (\partial_T)^a$;



$$\mathcal{L}_{\xi^a} g_{\mu\nu} = 0 .$$

$$(\text{i.e., } \nabla_a \xi_b + \nabla_b \xi_a = 0)$$

T plays a time coordinate associated with ξ^a;

$$\xi^a \nabla_a T = 1 .$$

$$ds^2 = -d\bar{u}d\bar{v} \quad , \quad \begin{cases} \bar{u} = T - X \\ \bar{v} = T + X \end{cases} . \qquad \Box \phi = 0$$

$$\phi \sim e^{-i\omega \bar{u}}, e^{-i\omega \bar{v}} \quad : \text{positive freq. modes} , \quad \mathcal{L}_{\partial_T} \phi = -i\omega \phi .$$

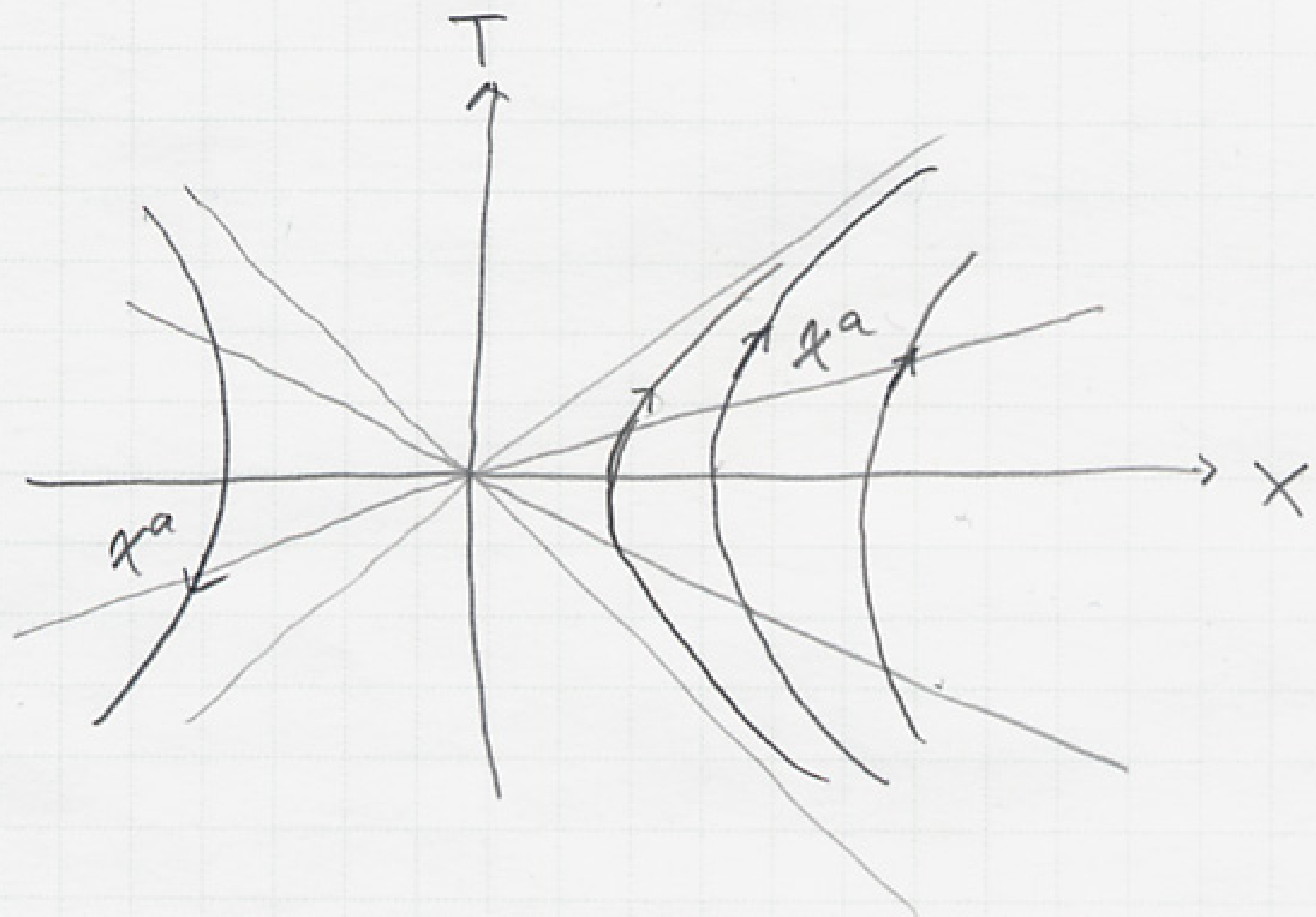
$\Rightarrow \quad |0\rangle_M$ in quantization

Note, however, that Minkowski spacetime admits other isometries as well.

$$\chi^a = a\left[X(\partial_T)^a + T(\partial_X)^a \right] :$$

$$\pounds_\chi g_{\mu\nu} = 0 \quad (\text{i.e., } \nabla_a \chi_b + \nabla_b \chi_a = 0)$$

This Killing vector field generates one-parameter group of Lorentz boost isometries.



$\Rightarrow$ Rindler spacetime.

* Rindler coordinates: $(T, X) \to (\eta, \xi)$

$$\begin{cases} T = a^{-1} e^{a\xi} \sinh a\eta \\ X = a^{-1} e^{a\xi} \cosh a\eta \end{cases} \qquad \begin{cases} \eta : (-\infty, \infty) \\ \xi : (-\infty, \infty) \end{cases}$$

w/ $a > 0$.

Or,
$$\begin{cases} \bar{u} = -a^{-1} e^{-au} \\ \bar{v} = a^{-1} e^{av} \end{cases} \quad \text{where} \quad \begin{cases} u = \eta - \xi \\ v = \eta + \xi \end{cases}$$

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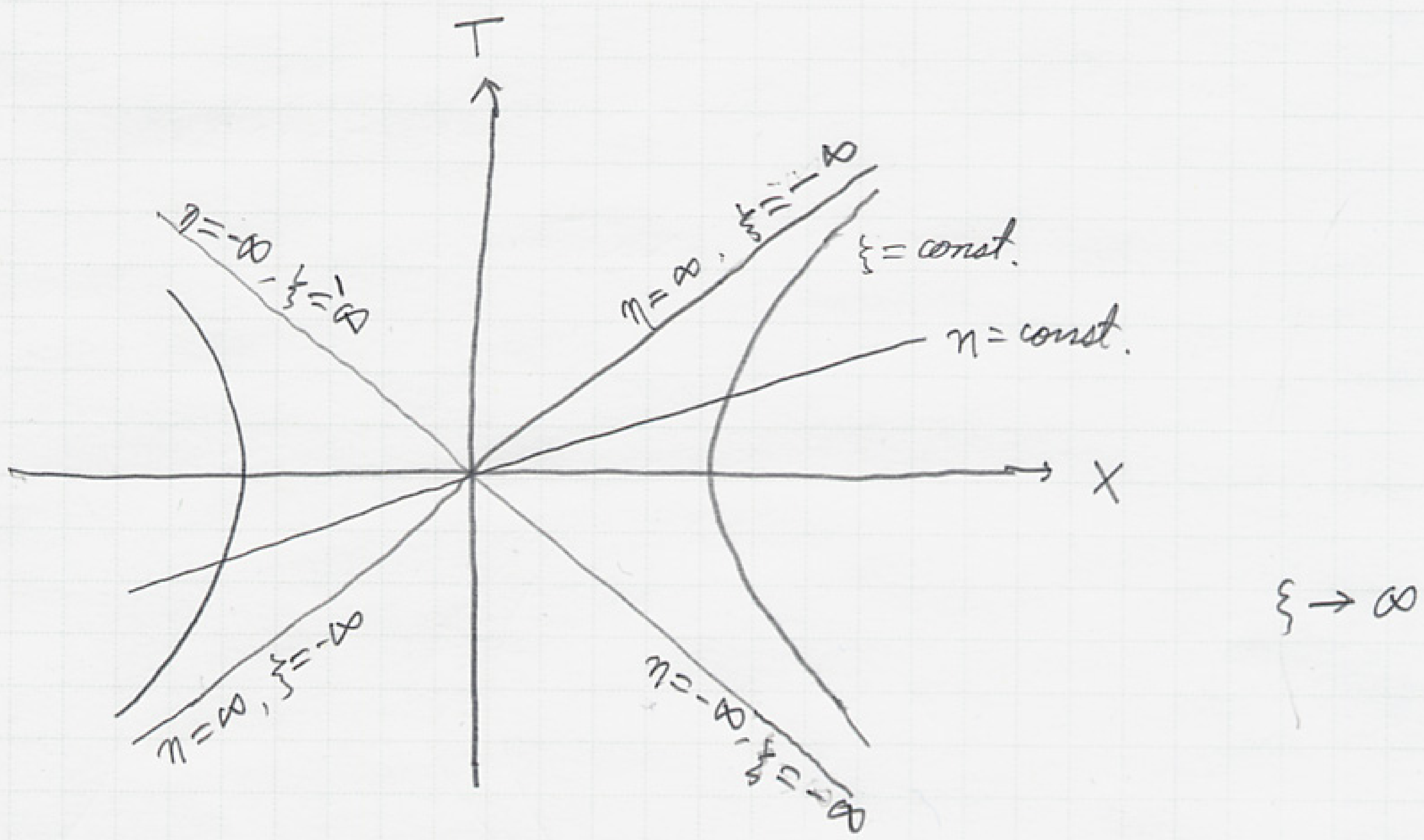
$\Rightarrow ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) = e^{2a\xi}\, du\, dv$.

Note

$$\begin{cases} \dfrac{T}{X} = \dfrac{\operatorname{sh} a\eta}{\operatorname{ch} a\eta} & \Rightarrow \ \eta = \text{const. are straight lines} \\[2ex] X^2 - T^2 = a^{-2} e^{2a\xi} & \Rightarrow \ \xi = \text{const. are hyperbolae} \end{cases}$$

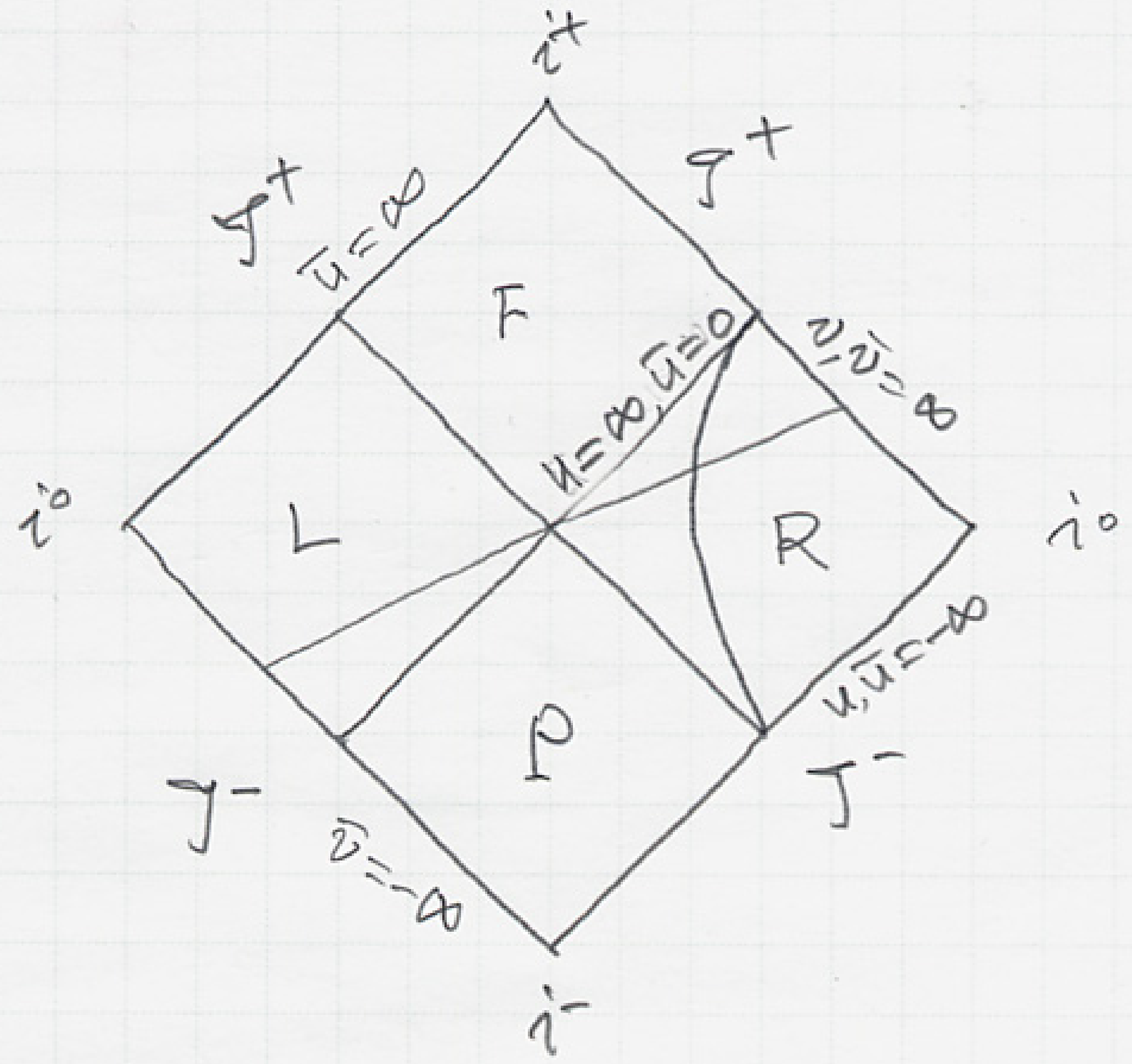


Indeed, $\xi = \text{const.}$ describes the world line of a uniformly accelerated observer having proper acceleration

$$\alpha^{-1} = a e^{-a\xi}$$

≡ Minkowski ST.

* Penrose diagram for the extended Rindler ST :



Its causal structure is very much similar to that of the Kruskal extension, (the extended Schwarzschild ST).

* Mode solutions : $\Box\phi = 0$.

$$\Box\phi = \left(-\frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial X^2}\right)\phi = \frac{\partial^2\phi}{\partial\bar{u}\,\partial\bar{v}} = 0 .$$

$$\Rightarrow \quad \bar{u}_k = \frac{1}{\sqrt{4\pi\omega}}\, e^{-i\omega T + ikX} \sim \begin{cases} e^{-i\omega\bar{u}} & k>0 \\ e^{-i\omega\bar{v}} & k<0 \end{cases} \qquad \omega = |k| .$$

$\mathcal{L}_{\partial_T}\bar{u}_k = -i\omega\,\bar{u}_k$: positive freq. modes w.r.t. the Killing v. ∂_T.

In the Rindler region,

$$e^{2a\xi}\,\Box\phi = \left(-\frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\xi^2}\right)\phi = \frac{\partial^2\phi}{\partial u\,\partial v} = 0$$

$$u_k = \frac{1}{\sqrt{4\pi\omega}}\, e^{\mp i\omega\eta + ik\xi} \sim \begin{cases} e^{-i\omega u} \\ e^{-i\omega v} \end{cases}$$

$-$ for R
$+$ for L.

$$\mathcal{L}_{\pm\partial_\eta} u_k = -i\omega\, u_k$$: positive freq. modes w.r.t. the timelike Killing vector ∂_η for R and $-\partial_\eta$ for L.

* Quantization

Define

$$^R u_k = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi} & \text{in } R \\ 0 & \text{in } L \end{cases}$$

$$^L u_k = \begin{cases} 0 & \text{in } R \\ \frac{1}{\sqrt{4\pi\omega}} e^{i\omega\eta + ik\xi} & \text{in } L . \end{cases}$$

$$\phi = \sum_{k=-\infty}^{\infty} \left(a_k \bar{u}_k + a_k^{+} \bar{u}_k^{*} \right)$$

$$= \sum_{k=-\infty}^{\infty} \left(b_k^{(1)}\, {}^L u_k + b_k^{(1)+}\, {}^L u_k^{*} + b_k^{(2)}\, {}^R u_k + b_k^{(2)+}\, {}^R u_k^{*} \right)$$

It yields two alternative Fock spaces with two vacuum states $|0\rangle_M$ or $|0\rangle_R$,

$\forall k$

$$a_k |0\rangle_M = 0$$

$$b_k^{(1)} |0\rangle_R = b_k^{(2)} |0\rangle_R = 0 .$$

* Bogolubov transformation:

$$\phi = \sum \left(b_k^{(1)}\, {}^L u_k + b_k^{(1)\dagger}\, {}^L u_k^* + b_k^{(2)}\, {}^R u_k + b_k^{(2)\dagger}\, {}^R u_k^* \right)$$

$$= \sum \frac{1}{\sqrt{2\,\mathrm{sh}\,\pi\omega/a}} \Big[d_k^{(1)} \left(e^{\pi\omega/2a}\, {}^R u_k + e^{-\pi\omega/2a}\, {}^L u_{-k}^* \right) + d_k^{(1)\dagger} \left(\quad\right)^*$$

$$+ d_k^{(2)} \left(e^{-\pi\omega/2a}\, {}^R u_{-k}^* + e^{\pi\omega/2a}\, {}^L u_k \right) + d_k^{(2)\dagger} \left(\quad\right)^* \Big]$$

positive freq. modes w.r.t. ∂_T.

$$d_k^{(1)} \propto \left(b_k^{(1)},\ b_k^{(2)},\ b_k^{(1)\dagger},\ b_k^{(2)\dagger} \right).$$

$$d_k^{(2)} \propto \left(\qquad 〃 \qquad \right)$$

$$\Rightarrow \begin{cases} b_k^{(1)} = \dfrac{1}{\sqrt{2\,\mathrm{sh}\,\pi\omega/a}} \left(e^{\pi\omega/2a}\, d_k^{(2)} + e^{-\pi\omega/2a}\, d_k^{(1)\dagger} \right) \\[2ex] b_k^{(2)} = \dfrac{1}{\sqrt{2\,\mathrm{sh}\,\pi\omega/a}} \left(e^{\pi\omega/2a}\, d_k^{(1)} + e^{-\pi\omega/2a}\, d_{-k}^{(2)\dagger} \right) \end{cases}$$

$|0\rangle_M$ is vacuum since the inertial observer detects no particle.

i.e., $${}_M\langle 0|\, a_k^\dagger a_k \,|0\rangle_M = {}_M\langle 0|\, d_k^{(1,2)\dagger} d_k^{(1,2)} \,|0\rangle_M = 0.$$

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However, the Rindler observer (i.e., a uniformly accelerated observer) detects particles,

$$ {}_M\langle 0 | \, b_k^{(1,2)\dagger} \, b_k^{(1,2)} \, | 0 \rangle_M = \frac{1}{e^{\omega/(a/2\pi)} - 1} $$

! ?

Moreover, this is precisely the Planck radiation at temperature

$$ T_0 = \frac{a}{2\pi} \,, $$

"THERMAL RADIATION"

the actual temperature seen by the accelerated observer is

$$ T = (-g_{00})^{-1/2} \, T_0 = (-x \cdot x)^{-1/2} \, T_0 \,. $$

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4. Hartle - Hawking state

Let $e^{-\pi\omega/a} = \mathrm{th}\,\phi_\omega$.

$$b_k^{(1)} = e^{iJ} d_k^{(1)} e^{-iJ} ,$$

where

$$J = \sum_k i\phi_\omega \left(b_{-k}^{(1)+} b_k^{(2)+} - b_{-k}^{(1)} b_k^{(2)} \right) .$$

$$|0\rangle_M = e^{-iJ} |0\rangle_R$$

$$= e^{\sum_k [-\ln \mathrm{ch}\,\phi_\omega + \mathrm{th}\,\phi_\omega\, b_k^{(1)+} b_k^{(2)+}]} |0\rangle_R$$

$$= \prod_k \frac{1}{\mathrm{ch}\,\phi_\omega} \sum_{n_k=0}^{\infty} e^{-n_k \pi\omega/a} |n_k^{(1)}\rangle |n_k^{(2)}\rangle$$

$$_M\langle 0| \hat{A} |0\rangle_M = \sum_{n_k} \prod_k \langle n_k^{(2)} | \hat{A} | n_k^{(2)} \rangle e^{-2n_k\pi\omega/a} [1 - e^{-2\pi\omega/a}]$$

$$= \mathrm{tr}(\hat{A}\rho) ,$$

where

$$\rho = \sum_n \prod_k \frac{e^{-\beta E_n}}{\sum_{m=0}^{\infty} e^{-\beta E_m}} |n_k\rangle\langle n_k|$$

with $E_n = n\omega$, $\beta = 2\pi/a$.

Comparing the case of the Kruskal extension with the present case, we see

$$|0\rangle_M \Longleftrightarrow |0\rangle_K$$

$$|0\rangle_R \Longleftrightarrow |0\rangle_S .$$

$$a \Longleftrightarrow \frac{1}{4M} = \kappa$$: surface gravity of the event horizon.

* <u>Euclidean Schwarzschild spacetime</u>:

$$R \equiv 4M\sqrt{1-\frac{2M}{r}} \;, \qquad \tau \equiv it$$

$$ds_E^2 = \left(1-\frac{2M}{r}\right)d\tau^2 + \frac{dr^2}{1-2M/r} + r^2 d\Omega^2$$

$$= R^2 d\left(\frac{\tau}{4M}\right)^2 + \left(\frac{r}{2M}\right)^2 dR^2 + r^2_{(R)}\, d\Omega^2 .$$

Coordinate singularity at $R=0$, i.e., $r=2M$ the horizon.

$\Rightarrow \exists$ Conical singularity in general.

It can be removed by giving a periodicity in τ

$$\frac{(\tau+\bar\beta)}{4M} = \frac{\tau}{4M} + \underbrace{\frac{\bar\beta}{4M}}_{=2\pi} \Longleftrightarrow \frac{\tau}{4M} \;\Rightarrow\; \boxed{T = \frac{\kappa}{2\pi}} .$$

$\frac{\kappa}{T} = 2\pi .$