Feynman Diagram & Planar limit

§ Feynman Diagram.

① Single variable

terms with odd-powers x^n identically vanish

$$\int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2} x A x} = (\sqrt{2\pi}) \cdot \frac{1}{A^{1/2}}$$

$$\int_{-\infty}^{\infty} dx \; x^2 \cdot e^{-\frac{1}{2} x A x} = (\sqrt{2\pi}) \cdot \frac{1}{A^{1/2}} \cdot \left[\frac{1}{A}\right]$$

$$\vdots$$

$$\int_{-\infty}^{\infty} dx \; x^{2n} \cdot e^{-\frac{1}{2} x A x} = (\sqrt{2\pi}) \cdot \frac{1}{A^{1/2}} \left[\frac{1}{A^n} \cdot \frac{(2n)!}{2^n n!} \right]$$

(i) useful to consider a following generating function.

LHS RHS

$$Z[b] \equiv \int_{-\infty}^{\infty} dx \underbrace{\left(\sum_{n=0}^{\infty} b_n x^n \frac{1}{n!} \right)}_{= e^{b \cdot x}} e^{-\frac{1}{2} x \cdot A \cdot x} = \sqrt{\frac{2\pi}{A}} \; e^{\frac{1}{2} b \cdot A \cdot b}$$

"check"

$$Z[b] = \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2} x \cdot A \cdot x + \underbrace{b \cdot x}}$$

"source term"

$$= \int_{-\infty}^{\infty} dx \quad e^{-\frac{1}{2}(x - bA^{-1}) \cdot A \cdot (x - bA^{-1}) + \frac{1}{2} b \cdot A^{-1} \cdot b}$$

indeed

$$= \underbrace{\left[\int_{-\infty}^{\infty} dx' \; e^{-\frac{1}{2} x' \cdot A \cdot x'} \right]}_{= \sqrt{\frac{2\pi}{A}}} \cdot \; e^{\frac{1}{2} b \cdot A^{-1} \cdot b}.$$

(ii) correlation function.

$$\langle x^2 \rangle_{(b=0)} \equiv \int_{-\infty}^{\infty} dx \; x^2 \cdot e^{-\frac{1}{2} x \cdot A \cdot x} \Big/ \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2} x A x}.$$

$e^{\frac{1}{2} b \cdot A^{-1} \cdot b}$

$$= \frac{1}{Z[0]} \left[\frac{\partial}{\partial b} \frac{\partial}{\partial b} Z[b] \right]_{b=0}.$$

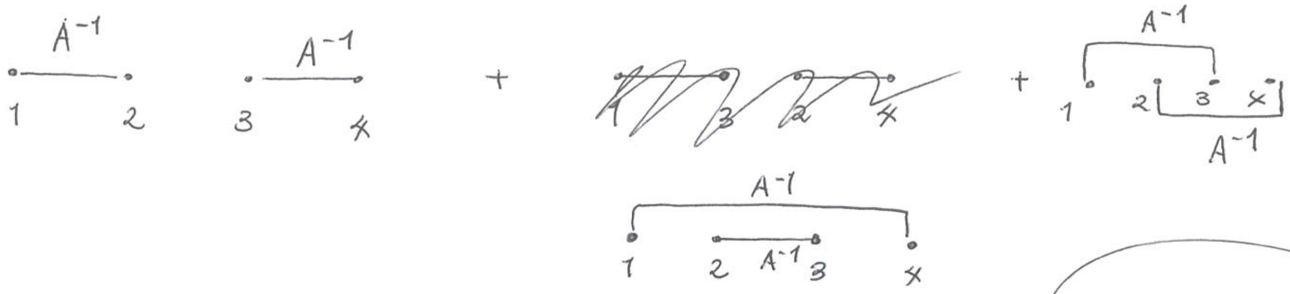
$$\underset{1 \qquad\qquad 2}{\bullet \overset{A^{-1}}{\text{―――――}} \bullet} \quad ; \quad \text{propagator}.$$

graphical rep.

$$\langle x^4 \rangle = \left(\frac{\partial}{\partial b}\frac{\partial}{\partial b}\frac{\partial}{\partial b}\frac{\partial}{\partial b} \cancel{\log} Z[b] \right)_{b=0} \cdot \frac{1}{Z[0]}$$

$$= \left(\frac{\partial^4}{\partial b^4} e^{\frac{1}{2} b\cdot A\cdot b} \right)_{b=0} .$$

$$= 3 \cdot \frac{1}{A^{-2}}$$



Wick thm

⋮

(iii) perturbation

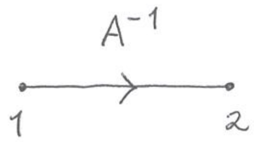
let's consider a <u>non-Gaussian</u> average

$$\int_{-\infty}^{\infty} dx \quad x^{2n} \quad e^{-\left(\frac{1}{2} x\cdot A\cdot x \; + \; g/_4 \, x^4\right)}$$

take the limit $g \to 0$ (asymptotic)

$$\int_{-\infty}^{\infty} dx \quad x^2 \cdot e^{-\frac{1}{2} x\cdot A\cdot x} \quad \Big(\underbrace{1}_{a} + \underbrace{\frac{(-g)}{4}x^4}_{b} + \underbrace{\frac{1}{2!}\cdot\left(\frac{-g}{4}\right)^2 x^8}_{c} + \cdots \Big)$$

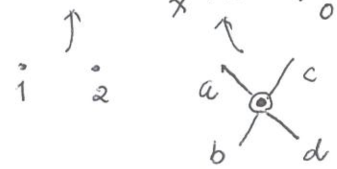
a ; $\underbrace{\sqrt{\frac{2\pi}{A}}}_{=\langle 1 \rangle_0} \cdot \underbrace{\frac{1}{A^{+1}}}_{=\langle x^2 \rangle_0}$



3 propagators

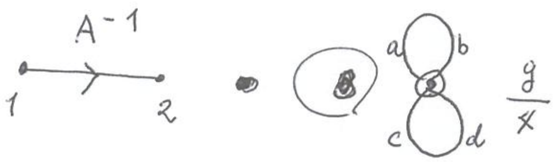
b ; $\underbrace{\sqrt{\frac{2\pi}{A}}}_{=\langle 1 \rangle_0} \cdot \frac{1}{A^3} \cdot \underbrace{\frac{6!}{2^3\cdot 3!}}_{=15}$

$\langle x^2 \cdot \frac{(-g)}{4} x^4 \rangle_0$



note that $15 = 3 + 12$,

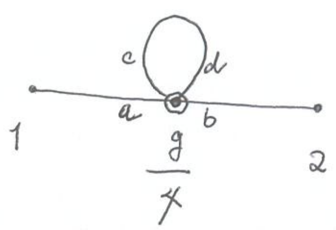
which allows the following graphical rep.



1 vertex.

3 $\quad \frac{4!}{2^2\cdot 2!} = 3$

+



12

$$\frac{4!}{2^1} = 12$$

Ⓒ ; $$\underbrace{\sqrt{\frac{2\pi}{A}}}_{\langle 1\rangle_0} \cdot \frac{1}{A^5} \cdot \underbrace{\frac{10!}{2^5 \cdot 5!} \cdot \frac{1}{2!}}_{=945} = \langle 1\rangle_0 \, \langle x^2 \cdot \frac{g}{4}x^4 \cdot \frac{g}{4}x^4 \rangle_0 \, \frac{1}{2!}$$

5-propagators

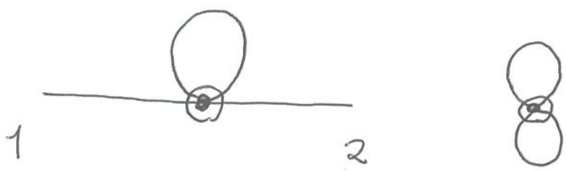
$\langle x^2 \cdot \frac{g}{4}x^4 \cdot \frac{g}{4}x^4 \rangle_0$:

"9"

two vertices

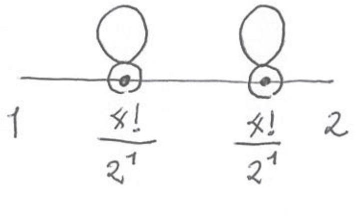
$\oplus \quad \frac{4! \times 4!}{4!} = $ "24"

$\frac{4!}{2^1} \times \frac{4!}{2^1} \times \frac{1}{2!} = $ "72"

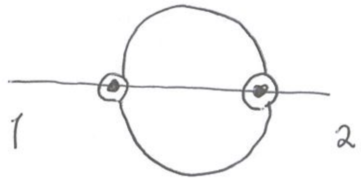


$2 \times 12 \times 3 = $ "72"

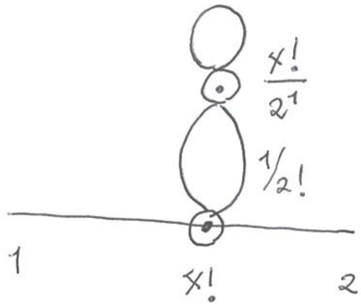
$\frac{4!}{2^1} = 12$ $\qquad$ $\frac{4!}{2^2 \cdot 2!} = 3$



$$2 \times \left(\frac{4!}{2^1}\right)^2 = \text{"}288\text{"}$$



$$2 \times \frac{4! \times 4!}{3!} = \text{"}192\text{"}$$



$$2 \times \left(4! \times \frac{4!}{2^1} \times \frac{1}{2!}\right) = \text{"}288\text{"}$$

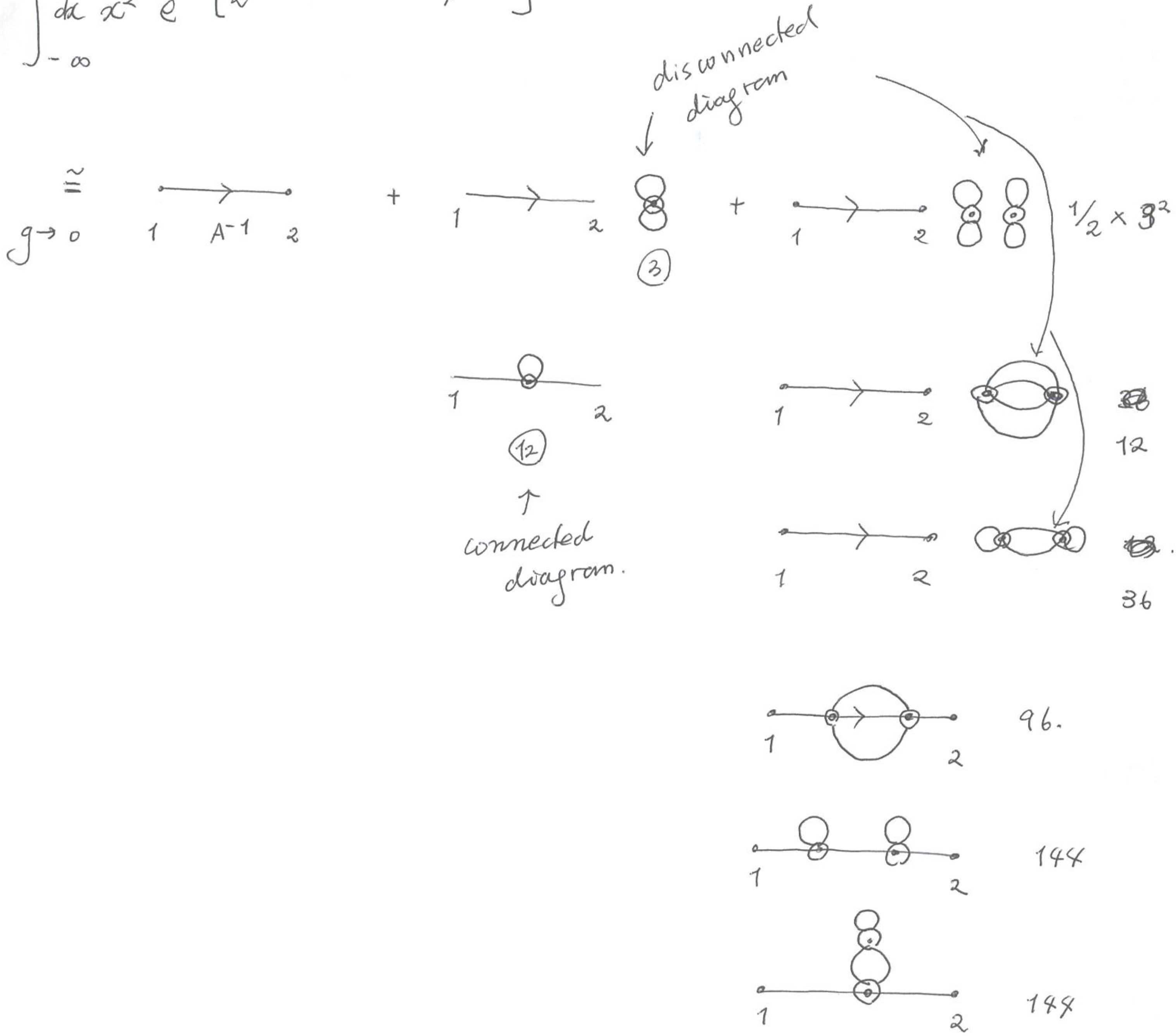
total number of combinations

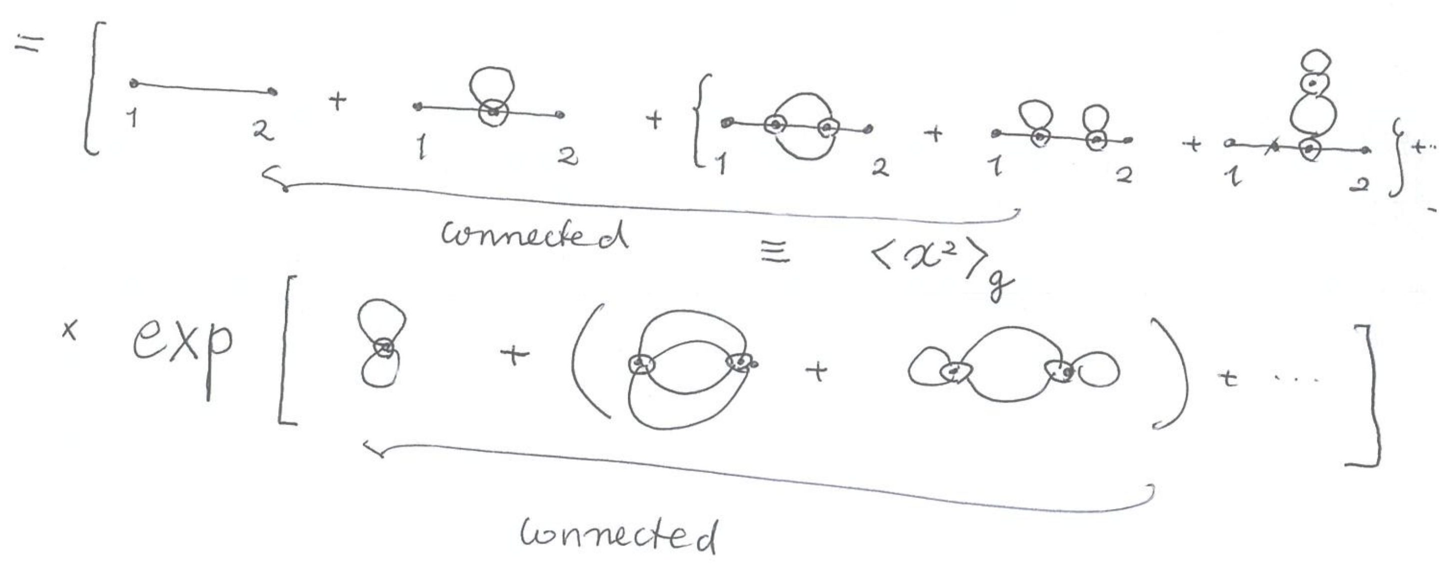
$= 9 + 24 + 72 + 72 + 288 + 192 + 288$

$= 945!$

Let's combine all the results.

$$\int_{-\infty}^{\infty} dx\, x^2\, e^{-\left[\frac{1}{2}\, x\cdot A\cdot x \;+\; \frac{g}{4} x^4\right]}$$



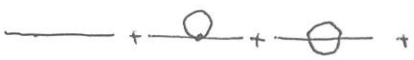


Hence, one can define the correlator as

$$\langle x^2 \rangle_g \equiv \frac{\int_{-\infty}^{\infty} dx \; x^2 \; e^{-\left[1/2 \; x\cdot A\cdot x \; + \; g/4 \; x^4 \right]}}{\underbrace{\int_{-\infty}^{\infty} dx \; e^{-1/2 \, x\cdot A\cdot x \, + \, g/4 \cdot x^4}}_{\equiv Z[g] \ \text{partition function}}}$$

(propagator)

$$\parallel$$

()

$$\equiv$$

$$A^{-1}(g)$$

$$= \langle 1 \rangle_{g=0} \cdot \exp\Big[\underbrace{\ \text{8} + \cdots\ }_{= \log\left(Z[g]/Z[0]\right)} \Big]$$

.... connected vacuum diagr

② Multi-variables

$$x \to x_i \to x_{ij} \to x_{ijk} \to \cdots$$

vectors matrices tensors

let's consider an integral below

$$\int \underbrace{[DM]}\, e^{-\frac{1}{2}\mathrm{tr} M^2} = \left(\sqrt{2\pi}\right)^{N^2}$$

Hermitian.

$$\equiv \underbrace{\prod_i dM_{ii} \left\{ \prod_{i<j} d\,\mathrm{Re}(M_{ij})\, d\,\mathrm{Im}(M_{ij}) \right\} 2^{N(N-1)/2}}$$

treat $M_i{}^j$ as an indepent real variable

where i & j are "unrestricted."

(i)

then $$\int DM \quad M_i{}^j M_k{}^l \, e^{-\frac{1}{2}\mathrm{tr} M^2} = ?$$

$$\int DM \quad M_i{}^j M_k{}^l M_m{}^n M_o{}^p \, e^{-\frac{1}{2}\mathrm{tr} M^2} = ?$$

to compute them, it is useful to compute the generating function by introducing the source term.

Hermitian

$$\cancel{\int [DM'] \, e^{-\frac{N}{2} \mathrm{tr}(M'^2 - 2SM')} \equiv I(S)}$$

rescale the matrix

$$M \to M/\sqrt{N} = M'$$

Hermitian

$$\propto \int [DM'] \; e^{-\frac{N}{2} \mathrm{tr}[M'^2] + \mathrm{tr}[SM']} = I(S)$$

rescale the matrix

$$M \to M/\sqrt{N} = M'$$

$$\because \; N/2 \left(M'^2 - 2N^{-1} S M' \right) = N/2 \, \left(M' - N^{-1} S \right)^2 + \frac{1}{2N} S^2$$

$$I(S) = \underbrace{(\cdots)}_{\text{indep. of } S} \times e^{\frac{1}{2} \frac{1}{N} \mathrm{tr}\, S^2}$$

$$\Rightarrow \langle M_{i_2}{}^{i_1} M_{j_2}{}^{j_1} \rangle_0 = \left[\frac{\partial}{\partial S_{i_1}{}^{i_2}} \frac{\partial}{\partial S_{j_1}{}^{j_2}} e^{\frac{1}{2}\frac{1}{N} \mathrm{tr} S^2} \right]_{S=0}$$

$$= \frac{1}{N} \delta^{i_1}{}_{j_2} \delta^{j_1}{}_{i_2}$$

Similarly, one can show that

$$\langle M_{i_2}{}^{i_1} M_{j_2}{}^{j_1} M_{k_2}{}^{k_1} M_{l_2}{}^{l_1} \rangle_0$$

$$= \frac{1}{N^2}\left\{ \delta^{i_1}{}_{j_2}\delta^{j_1}{}_{i_2}\,\delta^{k_1}{}_{l_2}\delta^{l_1}{}_{k_2} + \delta^{i_1}{}_{l_2}\delta^{l_1}{}_{i_1}\,\delta^{j_1}{}_{k_2}\delta^{k_1}{}_{j_2} + \delta^{i_1}{}_{k_2}\delta^{k_1}{}_{i_1}\,\delta^{j_1}{}_{l_2}\delta^{l_1}{}_{l_2} \right\}$$

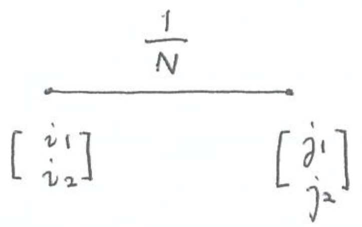
Wick's thm.

(ii) double-line graphs ('t Hooft)

ⓐ $\langle M_{i_2}{}^{i_1} M_{j_2}{}^{j_1} \rangle_0$ =

anti fundamental rep. index

fundamental rep. index



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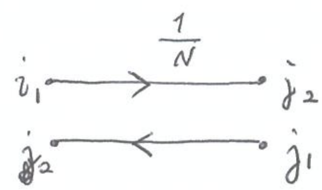
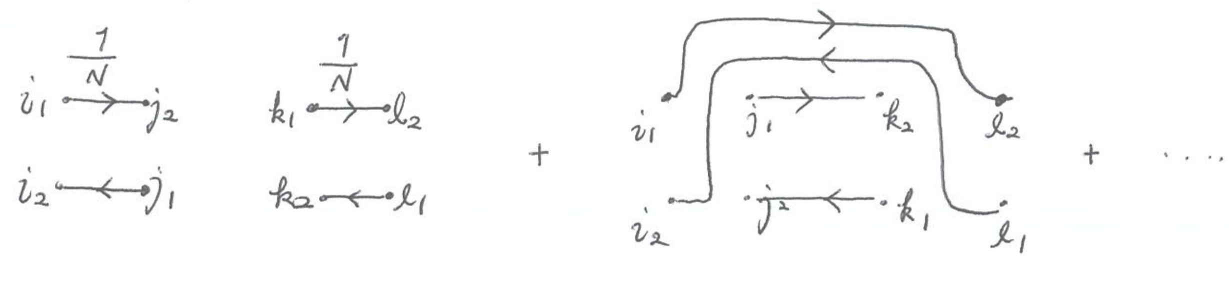


(arrow)
out-bound : lower
in-bound : upper

ⓑ $\langle M_{i_2}{}^{i_1} M_{j_2}{}^{j_1} M_{k_2}{}^{k_1} M_{l_2}{}^{l_1} \rangle_0$



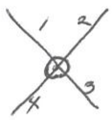
(iii) perturbation.

$$\int [DM] \; \cancel{e^{-\frac{1}{2}N\,\mathrm{tr}\,M^2}} \; e^{-N\left(\frac{1}{2}\mathrm{tr}M^2 + \underbrace{\frac{g}{\bullet}\mathrm{tr}M^4}_{\text{interaction}}\right)}$$

$$\underset{\substack{g\to 0\\ \text{limit}}}{\cong} \int [DM] \; e^{-\frac{N}{2}\mathrm{tr}M^2} \left(1 + \underbrace{\frac{(-g)}{\bullet}\mathrm{tr}M^4}_{ⓐ} + \underbrace{\cdots}_{ⓑ} \right)$$

ⓐ $M_i{}^j M_j{}^k M_k{}^l M_l{}^i$

ⓐ single variable :



$\left(-\frac{g}{\bullet}\right)$

vertex

$\left(-\frac{g}{\bullet}\right) \times 3$

$3 = 1 + 1 + 1$

(12) (13) (14)

(34) (24) (23)

$=(\alpha) = (\beta) = (\gamma)$

matrix :

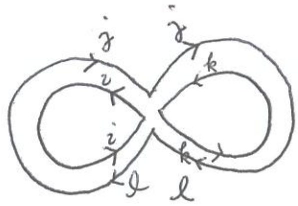
$\left(-\frac{g}{\bullet}\right) \times N$

(α) :

$\left(-\frac{g}{\bullet}\right) \times N$

double-line ~~rep.~~ graph

$$1 \to N \cdot \left(\frac{1}{N}\right)^2 \cdot N^3 = N^2$$

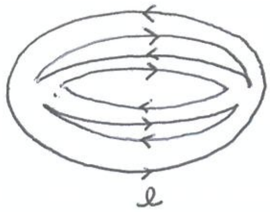
(γ) :  $1 \to N\cdot\left(\frac{1}{N}\right)^2\cdot N^3 = N^2$

$$\underbrace{\delta^i_i}_{=N}\ \underbrace{\delta^k_k}_{=N}\ \underbrace{\delta_l^{\,j}\delta_j^{\,l}}_{=N}$$

(β) :  $1 \to N\cdot\left(\frac{1}{N}\right)^2\cdot \underbrace{N}_{=\delta_i^{\,l}\delta_l^{\,k}\delta_k^{\,j}\delta_j^{\,i}} = 1$

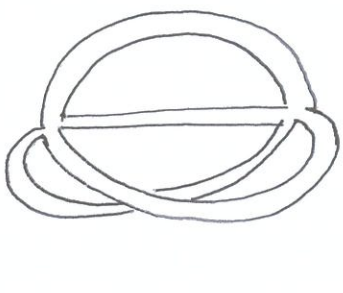
∴ first-order correction

$$\left(-\frac{g}{\ }\right)\cdot\left(2N^2+1\right)$$

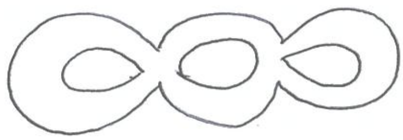
(b) :  ⇒  $2\cdot N^2\cdot\left(\frac{1}{N}\right)^4\cdot N^4 = 2N^2$

12
=
2 + 10

$\left(-\frac{g}{\ }\right)^2$

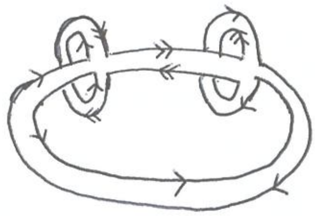
 $10\cdot N^2\cdot\left(\frac{1}{N}\right)^4\cdot N^2 = 10$

$\Rightarrow$

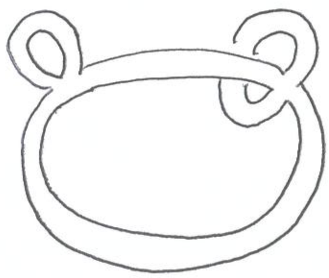


$36 = 16 + 4 + 16$

$16 \cdot N^2 \cdot \left(\frac{1}{N}\right)^4 \cdot N^4 = 16 N^2$



$4 \cdot N^2 \cdot \left(\frac{1}{N}\right)^4 \cdot N^2 = 4$



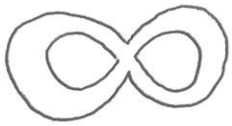
$16 \cdot N^2 \cdot \left(\frac{1}{N}\right)^4 \cdot N^2 = 16.$

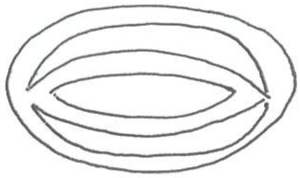
(iv) large N & topological expansion

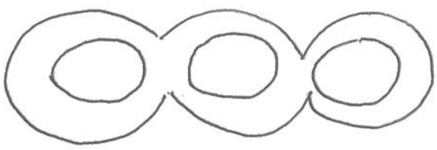
$$Z_N(g) / Z_N(0) = \exp\left[\text{connected diagram}\right]$$

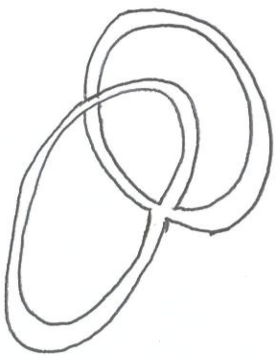
large N-limit

$$= N^2 \left\{ \underset{ⓐ}{\cdots} \right\} + N^0 \left\{ \underset{ⓑ}{\cdots} \right\} + N^{-2} \left\{ \cdots \right\}$$

ⓐ :  $\left(-\frac{g}{\cancel{N}}\right)\cdot 2$

 $\left(-\frac{g}{\cancel{N}}\right)^2\cdot 2$

 $\left(-\frac{g}{\cancel{N}}\right)^2\cdot 16.$

ⓑ   $\left(-\frac{g}{\cancel{N}}\right)\cdot 10$

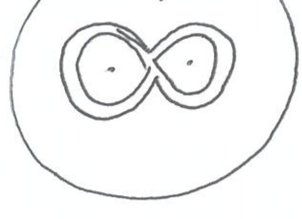
...

$\vdots$

$\exists$ $\underbrace{\text{characteristic difference}}_{}$ btw ⓐ, ⓑ, and so on.

topological difference

ⓐ : sphere diagram



$N^V (N^{-1})^E (N)^F$

Ⓥ: number of vertices

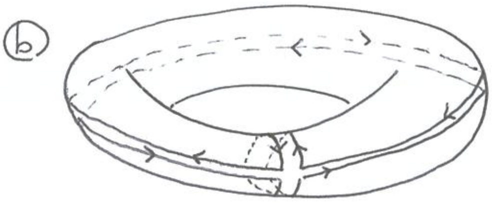
Ⓔ: number of edges (double-li…

Ⓕ: number of loops (= faces)

$$N^{V-E+F} = (N)^{\chi(M_2)} = (N)^{+2}$$

Euler characteristic of S^2

(b)

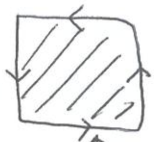


$$N^{V-E+F} = N^{1-2+1} = N^{0}$$

$$= N^{\chi(\ )}$$

~~note that~~ $\cancel{\chi(\ ) = 2-2g}$

? F = loop ?



face

$$\delta_k{}^{j}\,\delta_j{}^{i}\,\delta_i{}^{l}\,\delta_l{}^{k}$$

note that

$$\chi(\ ) = 2(1-h)$$

number of genus.

$$\log \left[Z_N(g) \Big/ Z_N(0) \right] = - E_N(g)$$

$$\|$$

connected diagram $\overset{\text{large } N}{=} \sum_h N^{2-2h} \underbrace{E^h(g)}_{\text{series expansion of } g}$

double expansion.

One can show that

$$\begin{cases} E^{h=0}(g) = 2g - 18g^2 + 288g^3 - 6048g^4 + \cdots \\ E^{h=1}(g) = g - 30g^2 + 1056g^3 - 40176g^4 + \cdots \\ E^{h=2}(g) = 240g^3 - 32112g^4 + \cdots \\ \vdots \end{cases}$$

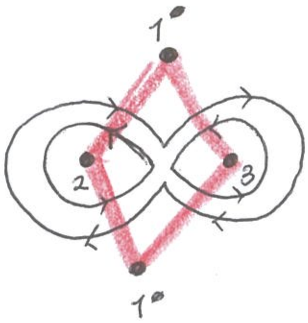
remark in order to make the double expansion well-defined,

the series $E_N^h(g)$ are convergent & $E_N^h(g)$ decreases as N increases.

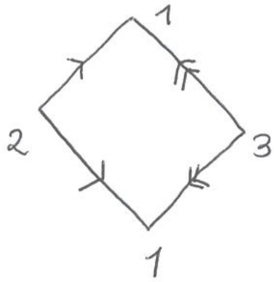
safe limit $N \to \infty$ & $E_N^h(g)$ convergent as $g \to 0$

(V) discretized surface = " dual graph "
(quadrangulation)

ⓐ sphere ~ planar diagram.



dual graph



~ sphere.

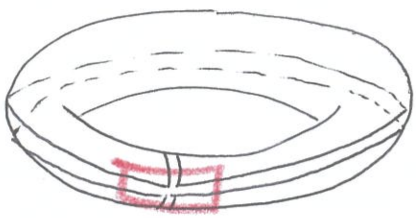
$\tilde{F} = 1$

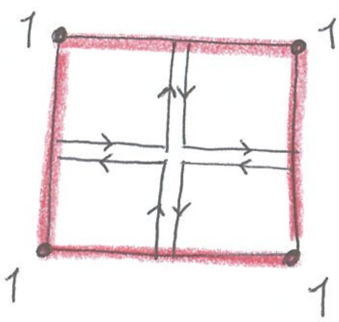
$\tilde{E} = 2$

$\tilde{V} = 3$

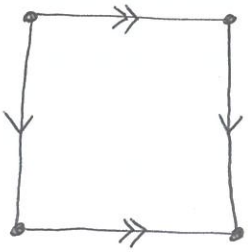
$\chi(\tilde{M}) = 1 - 2 + 3$
$= 2$

ⓑ torus





dual graph



$\tilde{F} = 1$

$\tilde{E} = 2$

$\tilde{V} = 1$

~ torus.

(c)

$$\log Z[g]/Z[0] \quad \boxed{\text{connected diagram of RMT}}$$

$$\underset{\text{large } N}{=} \sum_{\text{topology}} N^{2(1-h)} \sum_{n=1}^{\infty} a_{h,n}\, g^{n}$$

number of vertices $\Leftarrow$ n = number of quadrangles !

$$\boxed{\text{"} = Z_{\text{quantum gravity}} \text{"}}$$

$\boxed{Q}$ ~~remarks~~ (i) $\sum_{n=1}^{\infty} a_{h,n}\, g^{n}$ convergent ?

(ii) $\sum_{h} N^{2(1-h)} a_{h}(g)$ convergent ?

to answer the question (i), we use the saddle-point approximation.