Large N Limit & Saddle point approximation

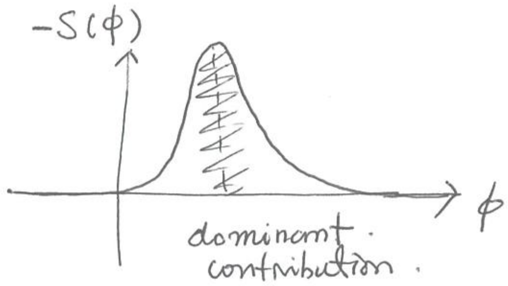
$$(\mathrm{const})_N \int \prod_i d\lambda_i \; e^{-\left[\frac{1}{2}\cancel{g}\sum_i \lambda_i^2 + \frac{g}{N}\sum_i \lambda_i^4\right]} \prod_{i<j} |\lambda_i - \lambda_j|^2$$

Dyson GAS

$$\to (\mathrm{const}')_N \int \prod_i d\lambda_i' \; e^{\underbrace{-N\sum_{i=1}^{N}\left(\frac{\lambda_i'^2}{2} + g\lambda_i'^4\right)}_{\text{order } N^2} + \underbrace{\sum_{i\neq j} \log|\lambda_i - \lambda_j|}_{\text{order } N^2.}}$$

rescale
the eigenvalue $\lambda_i' = \lambda_i / \sqrt{N}$ remark (normalize the range of eigenvalues)

§ large N limit.



(i) Saddle-point approximation

$$\int_{-\infty}^{\infty} d\phi \; e^{-\frac{1}{\hbar}S(\phi)} \underset{\hbar\to 0.}{\cong} \int^{\infty} d\delta\phi \; e^{-\frac{1}{\hbar}\left[S(\phi_0) + \frac{1}{2}S''(\phi_0)\delta\phi^2 + \cdots\right]}$$

$+ \; O(\delta\phi^3)$

$$S(\phi) = S(\phi_0 + \delta\phi) \cong S(\phi_0) + \frac{1}{2}\delta\phi^2 S''(\phi_0)$$

where $\underset{\uparrow\ \text{saddle-pt}}{S'(\phi = \phi_0) = 0}$.

rescale $\delta\phi^2/\hbar \equiv \underset{\uparrow\ \text{finite.}}{y^2}$

$$\underset{\hbar\to 0}{\simeq} \left[\int_{-\infty}^{\infty} \hbar\, dy \quad e^{+\left(\frac{1}{2}S''(\phi_0)y^2 + \underbrace{\circ\hbar y^3 + \circ\hbar^2 y^4 + \cdots}_{\hbar\text{ - correction}}\right)} \right] \cdot e^{+\frac{1}{\hbar}S(\phi_0)}$$

$$\cong (\text{const.}) \; \underbrace{e^{+\frac{1}{\hbar}S(\phi_0)}}_{\text{"leading term."}} \sqrt{\frac{2\pi}{-S''(\phi_0)}} \left(1 + \sum_{n=1}^{\infty} \hbar^n a_n\right).$$

$$\sim \exp\left[+ \left(\frac{1}{\hbar} S(\phi_0) + \frac{\hbar^0}{2} \log[-S''(\phi_0)] + \sum_n \hbar^n b_n \right) \right]$$

(ii) in the large N-limit, one can use the saddle approximation to evaluated the integral.

$$\int d\lambda_i \quad e^{-N S(\lambda_i)} \qquad \text{where } S(\lambda_i) = \frac{1}{2}\sum_i \lambda_i^2 + g \sum_i \lambda_i^4 + \underbrace{\frac{1}{N} \sum_{i \neq j} \log|\lambda_i - \lambda_j|}_{\text{the same order of } N}$$

ⓐ Saddle-point equation $S'(\lambda_i = \tilde{\lambda}_i) = 0$.

$$\Leftrightarrow \quad \boxed{\tilde{\lambda}_i + 4g\tilde{\lambda}_i^2 = \frac{2}{N} \sum_{j \neq i} \frac{1}{\tilde{\lambda}_i - \tilde{\lambda}_j}}.$$

∵ assume that $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_N$

(b) ~~In the large~~ in the large N limit, one can solve the saddle-point equation by taking the continuum limit.

increasing function of x.

$$\begin{cases} \tilde{\lambda}_i \Rightarrow \lambda(x) \quad \text{where } x = \frac{i}{N} \quad x \in (0, 1) \\ \qquad\quad\ \ \| \\ \qquad\quad\ i/N \\ \frac{1}{N}\sum_i \Rightarrow \int_0^1 dx \end{cases}$$

then, the equation becomes.

$$\frac{1}{2}\lambda(x) + 2g\lambda^3(x) = \underbrace{⨍_0^1 dy}_{\text{"principal value at } y=x\text{"}} \frac{1}{\lambda(x) - \lambda(y)}$$

(c) change of variable from $x \to \lambda$

$\Rightarrow$ density of eigenvalues $\quad dx = d\lambda\, \rho(\lambda)$.

assumption $\boxed{\text{one-cut}} \Rightarrow \lambda \in (-2a, 2a)$

end-point

thus,

$$\underbrace{\left(\frac{1}{2}\lambda + 2g\lambda^3 \right)}_{1/2\, V'(\lambda)} = \underbrace{⨍_{-2a}^{2a}}_{\substack{\text{principal} \\ \text{value at } \mu=\lambda.}} d\mu \, \frac{\rho(\mu)}{\lambda - \mu}$$

$\boxed{Q}$ how to determine $\rho(\lambda)$ & a ?

⓪ useful to define a function $\omega(\lambda)$ below

$$\omega(\lambda) = \int_{-2a}^{2a} d\mu \, \frac{\rho(\mu)}{\lambda - \mu} \qquad \text{(resolvant)}$$

stieltjes integral

$$\left\{ \begin{array}{ll} (\alpha) & \omega(\lambda) \text{ is analytic away from } (-2a, 2a). \\ (\beta) & |\lambda| \to \infty . \quad \omega(\lambda) \to \frac{1}{\lambda} . \left(\because \int_{-2a}^{2a} d\mu \, \rho(\mu) = 1 \right) \\ (\gamma) & \omega(\lambda) \text{ is real if } \lambda \text{ is real } \& \ \lambda \notin (-2a, 2a) \\ (\delta) & \omega(\lambda + i\epsilon) - \omega(\lambda - i\epsilon) \neq 0 \end{array} \right.$$

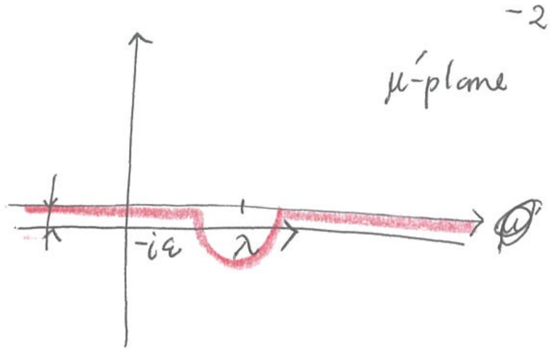
$\exists$ branch cut when $\lambda \in (-2a, 2a)$

$\varepsilon \to 0^+$

$$\omega(\lambda+i\varepsilon) = \int_{-2a}^{2a} d\mu \, \frac{\rho(\mu)}{\lambda+i\varepsilon-\mu}$$

no discontinity

$$= \int_{-2a\mp i\varepsilon}^{2a\mp i\varepsilon} d\mu' \, \frac{\overbrace{\rho(\mu'+i\varepsilon)}^{\sim\,\rho(\mu')}}{\lambda-\mu'}$$



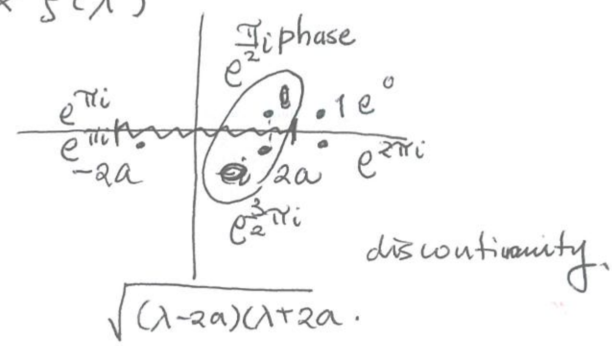
$$\Rightarrow \fint_{-2a}^{2a} d\mu' \, \frac{\rho(\mu')}{\lambda-\mu'} - i\pi\rho(\lambda).$$

Similarly, one can show $\omega(\lambda-i\varepsilon) = \underbrace{\fint_{-2a}^{2a} d\mu' \, \frac{\rho(\mu')}{\lambda-\mu'}}_{=\left(\frac{1}{2}\lambda+2g\lambda^3\right)} + i\pi\rho(\lambda)$

from (δ), one can verify that (saddle-pt eqn.)

$$\omega(\lambda) = \left(\frac{1}{2}\lambda + 2g\lambda^3\right) - \underbrace{\overset{\text{analytoc}}{f(\lambda)}\sqrt{(\lambda-2a)(\lambda+2a)}}_{\propto\,\rho(\lambda)}$$

one can fix $f(\lambda)$ by the condition (β)



$$\omega(\lambda) = \left(\frac{1}{2}\lambda + 2g\lambda^3\right) - \left(\frac{1}{2} + 4g\,a^2 + 2g\lambda^2\right)\sqrt{\lambda^2 - 4a^2}$$

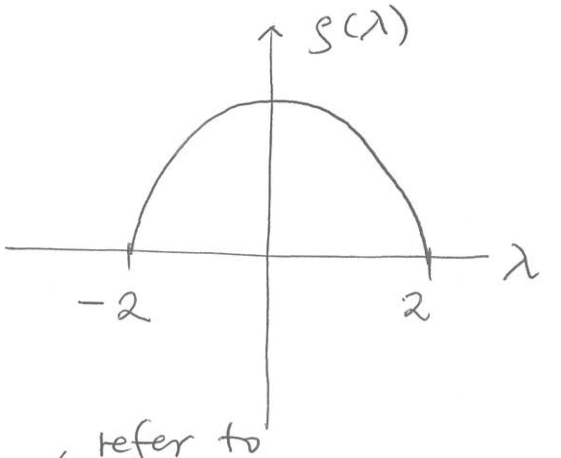
where $12ga^4 + a^2 - 1 = 0$, i.e $\boxed{a_{(g)}^2 = \frac{-1+\sqrt{1+48g}}{24g}}$.

$$\Rightarrow \quad \rho(\lambda) = \frac{1}{\pi}\left(\frac{1}{2} + 4g\,a_{(g)}^2 + 2g\lambda^2\right)\sqrt{(2a_{(g)}-\lambda)(\lambda+2a_{(g)})} \qquad \lambda \in (-2a, 2a)$$

ⓔ density of eigenvalues :

Wigner semi-circle law ($g=0$. Gaussian).

$$\rho(g=0, \lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}$$



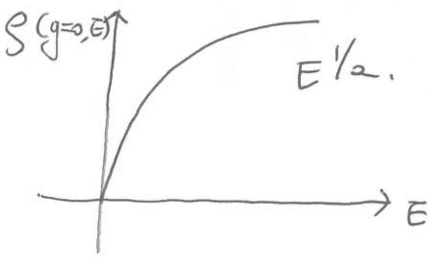
(refer to numerical result)

near $\lambda = 2$,

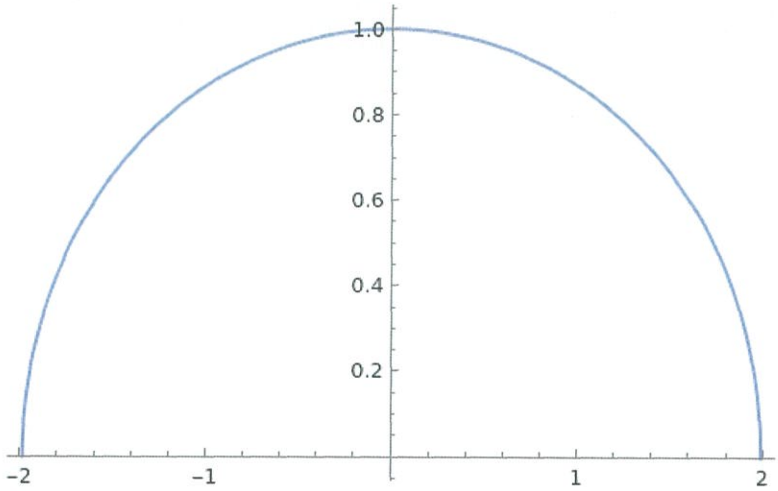
~~near~~ $\lambda \cong \cancel{2(1-E)}\ 2-E$ small

$\cancel{\rho(E)}$

$$\rho(g=0, E) = \frac{1}{\pi} E^{1/2}$$

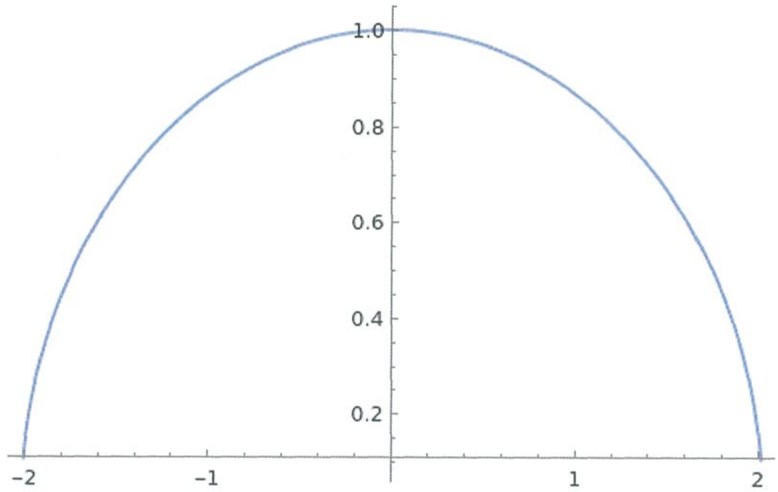


$$\text{Plot}\left[\rho[\lambda, 0.001], \left\{\lambda, -2\sqrt{a[0.001]}, 2\sqrt{a[0.001]}\right\}\right]$$

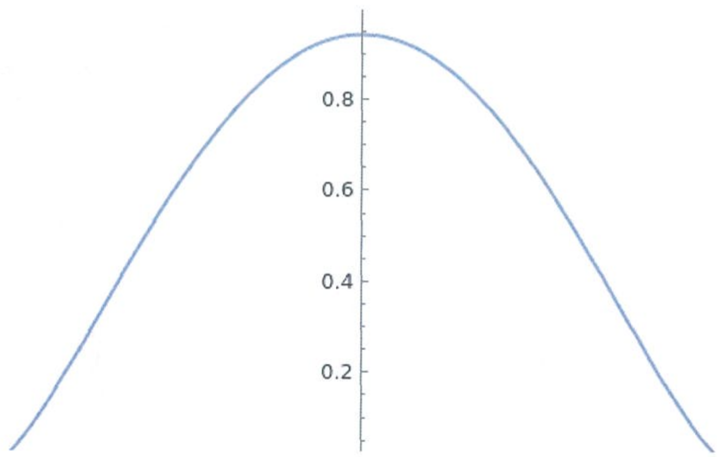


analytic continuation possible!

$$\text{Plot}\left[\rho[\lambda, -0.001], \left\{\lambda, -2\sqrt{a[-0.001]}, 2\sqrt{a[-0.001]}\right\}\right]$$

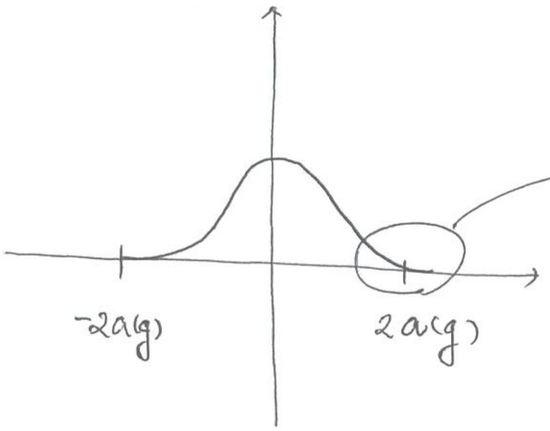


$$\text{Plot}\left[\rho\left[\lambda, \frac{-1}{48}\right], \left\{\lambda, -2\sqrt{a\left[\frac{-1}{48}\right]}, 2\sqrt{a\left[\frac{-1}{48}\right]}\right\}\right]$$



$$\rho\left(g=-\frac{1}{48},\lambda\right) = \frac{1}{\pi}\underbrace{\left(\frac{1}{2} - \frac{4}{48}\cdot 2 - \frac{2}{48}\lambda^2\right)\sqrt{(2\sqrt{2}-\lambda)(\lambda+2\sqrt{2})}}$$

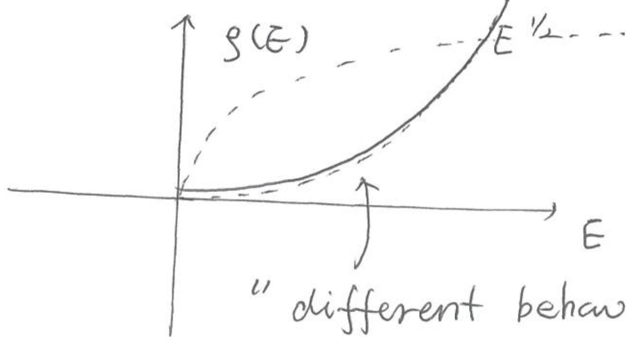
$$= \frac{(8-\lambda^2)}{24}\sqrt{8-\lambda^2}$$



near $\lambda^2 = 8$, i.e,

$\lambda \cong 2\sqrt{2} - E$

$\rho(E) \propto E^{\frac{3}{2}}$



" different behavior "

when $g \to -\frac{1}{48} \equiv g_c$

critical value

(f) leading term in the large N limit

$$e^{-NS(\tilde{\lambda}_i)} = e^{-N\underbrace{\left[\sum_i\left(\frac{1}{2}\tilde{\lambda}_i^2 + g\tilde{\lambda}_i^4\right) - \frac{1}{N}\sum_{j\neq i}\log|\tilde{\lambda}_i - \tilde{\lambda}_j|\right]}}$$

$$[\cdots] \simeq N\left\{\int dx\,\left(\frac{1}{2}\lambda^2(x) + g\lambda^4(x)\right) - \int dx\int dy\,\log|\lambda(x)-\lambda(y)|\right\}$$

$$\therefore\quad e^{-N^2\left\{\int_0^1 dx\,\left(\frac{1}{2}\lambda^2(x)+g\lambda^4(x)\right)-\int_0^1 dx\fint_0^1 dy\,\log|\lambda(x)-\lambda(y)|\right\}}$$

change of variable $dx = d\lambda\cdot\rho(\lambda)$

$$\int_{-2a}^{2a} d\lambda\,\rho(\lambda)\left(\frac{1}{2}\lambda^2+g\lambda^4\right)-\int_{-2a}^{2a} d\lambda\fint_{-2a}^{2a} d\mu\;\rho(\lambda)\rho(\mu)\log|\lambda-\mu|$$

from the saddle-pt eqn. one can show that

integral over λ

$$\cancel{\int\left(\frac{1}{2}\lambda^2+g\lambda^4\right)}$$

$$\frac{1}{2}\lambda^2+g\lambda^4 = 2\fint_{-2a}^{2a} d\mu\;\rho(\mu)\log|\lambda-\mu| + \underline{\text{integration const in }\lambda}$$

$\lambda = 0$

$$0 = 2\fint_{-2a}^{2a} d\mu\,\rho(\mu)\log|\mu| + \text{const in }\lambda$$

principal value at $\mu=0$

$\Rightarrow$ the term in $\{\;\cdots\;\}$ becomes

no difference between $\fint$ & $\int_{pr.}$

$$\int_{-2a}^{2a} d\lambda\;\rho(\lambda)\left(\frac{1}{4}\lambda^2+\frac{g}{2}\lambda^4\right)-\fint_{-2a}^{2a} d\mu\;\rho(\mu)\log\mu$$

$$\underbrace{\log Z[g]/Z[0]}_{= Z_{\text{gravity}}} = -N^2\left[\int_{-2a}^{2a} d\lambda\, \rho(\lambda)\left(\frac{1}{4}\lambda^2+\frac{g}{2}\lambda^4-\log|\lambda|\right) - (g=0)\right] + (\cdots)$$ ⑨

$$= +N^2\left[\frac{1}{2}\log a^2(g) - \frac{1}{24}\left(a^2(g)-1\right)\left(9-a^2(g)\right)\right] + (\cdots)$$

$$\left[\cdots\right] = -E^{h=0}$$

$$a^2(g) = \frac{-1+\sqrt{1+48g}}{24g}$$

<u>remember that</u> $\log Z[g]/Z[g=0] = -\sum_h N^{2-2h} E_N^h(g)$

$$E^0(g) = \frac{1}{24}\left(9-a^2(g)\right)\left(a^2(g)-1\right) - \frac{1}{2}\log a^2(g)$$ //

$$\underset{g\to 0}{\simeq}\ 2g - 18g^2 + 288g^3 - 6048g^4 + \cdots$$

convergent series!

perfect agreement with the result obtained by the Feynman diagram method remark why convergent series rather than asymptotic series?

A : due to double expansion in term of $\frac{1}{N}$ & g

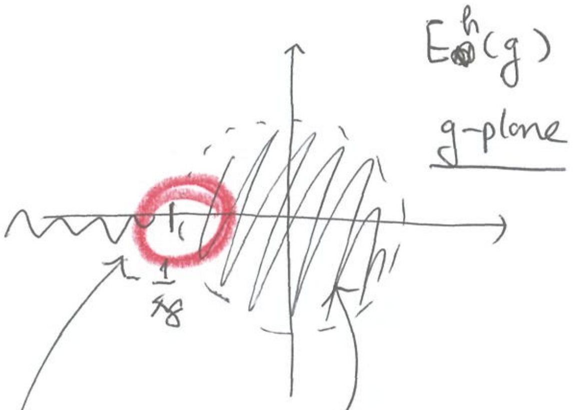
$$E(g) = \sum_h N^{2-2h} \underbrace{E^h(g)}$$

$\Rightarrow$ convergent series of g

$\Rightarrow$ asymptotic series of g

$\boxed{Q}$ radius of convergence $E^h(g)$?

analytic continuation possible from $g>0$ to $g<0$ until $g \to g_c = -\frac{1}{48}$



※ since $E^h(g)$ is a convergent series, one can have a small but finite value of g

qualitatively different behaviors $\Rightarrow$ ① Continuum limit of quantum gravity. ② all genus expansion is possible even when $N \to \infty$

§ double scaling limit

$$48g \sim (-1 + h)$$

very small.

$$a^2(h) = \frac{2}{1+\sqrt{h}} \cong 2(1 - \sqrt{h} + h + \cdots)$$

then, one can obtain $\quad E = \sum_{h=0}^{\infty} (N^2)^{2-2h} \cdot E^h(g)$

$$E^{h=0}(g) = \frac{1}{24}(a^2(g) - 1)(9 - a^2(g)) - \frac{1}{2}\log a^2(g)$$

$$\simeq \underbrace{\left(\frac{7}{24} - \frac{1}{2}\log 2\right) + \frac{1}{12}h - \frac{1}{8}h^2} + \frac{4}{15}(\sqrt{h})^5 + \cdots$$

$$E^{h=1}(g) \sim \# \log\sqrt{h}$$

$$E^{h>1}(g) \sim \# (\sqrt{h})^{5(1-h)}$$

remnants of the cut-off

they are <u>not universal</u> which means that the precise form of these terms depends on the model

In summary,

$$E = N^2\left((\cdots) + \frac{4}{15}(\sqrt{h})^5 + \mathcal{O}(h^3)\right) + N^0\left(\log\sqrt{h} + \cdots\right)$$

$$+ \sum_{h'} N^{-2(h'-1)}\left(\beta(\sqrt{h})^{5(1-h')} + \text{higher orders in } \sqrt{h'}\right)$$

"strictly speaking, in the naive $N \to \infty$ limit, all contributions except the planar diagrams are gone!

$\boxed{\text{double scaling limit}}$

① $N = \dfrac{1}{\underbrace{\hbar}_{\text{finite}} \underbrace{\alpha^5}_{\substack{\alpha \to 0 \\ \text{small}}}} \to \infty$ (i.e. $N\alpha^5$ = finite)

② $\hbar = \underbrace{l}_{\text{finite}} \cdot \alpha^4 \to 0$ (i.e., $\hbar \cdot \alpha^{-4}$ = finite)

※ these terms are not universal, but depend on the model. (~ scheme dependence) RG

then,

$$E \to \frac{1}{\hbar^2}\Big[\underbrace{(\cdots)}_{\substack{\text{divergent} \\ \text{"cut-off terms"}}} + \underbrace{\frac{\#}{15}(\sqrt{l}\,)^5 \Big] + \hbar^0 \cdot \overset{\#}{\log}\sqrt{l} + \sum_{n=2}^{\infty} \hbar^{-2+2n} \cdot (\beta \sqrt{l}\,)^{5(1-n)}}_{!\ \boxed{\text{universal}}\ !}$$

continuum limit of discretized surface

"def. of quantum gravity"!! where

$\hbar^{-1}$: string coupling

$l \sim$ cosmological const $\qquad \cancel{\Lambda^{ren} \sim g_c (g - g_c) / \alpha^4}$

$$e^{\left[-\Lambda^{ren} \times \binom{\text{infinitesimal}}{\text{unit area}}\right]} \simeq g/g_c .$$

why continuum limit?

$$g \to g_c = -1/48 \qquad \left(48\, g = 48 g_c + l \cdot \alpha^4 \right)$$

$$Z^{S^2}_{\text{gravity}} \sim (g - g_c)^{\frac{5}{2}} \qquad \text{where } g \sim e^{\left[-\frac{\Lambda}{8\pi} (\text{Area})\right]}$$

bare cosmological const

area of each quadrangle

$$= \sum_{\text{lattice}} a_V g^V$$

$\cancel{\frac{\partial}{\partial g} Z^{S^2}_{\text{gravity}}(g)}$

⟨Area of surface S^2⟩ $\propto$

$$\langle V \rangle = \frac{\partial}{\partial g} \log Z^{S^2}_{\text{gravity}}(g) \sim \frac{1}{g - g_c}$$

"average" numbers of vertices for quadrangulation of given surface

Hence, in $g \to g_c$ limit number of vertices of dual graphs (or equivalently # of quadrangles) diverges!!

Continuum limit!

Take-Home Message $\qquad$ double-scaling limit

① universal behavior (model-independent)

② all-genus expansion

③ continuum limit